Proof of the Rokhlin's Conjecture on Arnold's surfaces

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Abstract. In this paper we prove that Arnold Surfaces of all real algebraic curves of even degree with non-empty real part are standard (Rokhlin's Conjecture). There is an obvious connection with classification of Arnold Surfaces up to isotopy of S^4 and Hilbert's Sixteen Problem on the arrangements of connected real components of curves. First, we consider some M-curves, i.e curves of a prescribed degree having the greatest possible number of connected real components, and prove that Arnold surfaces of these curves are standard. Afterwards, we define a procedure of modification "perestroika" of these M-curves which allows to prove the Rokhlin's Conjecture.

Contents

Part 1

Introduction

A real algebraic curve $\mathcal{A}$ of even degree is by definition a homogeneous real polynomial $F(x_0, x_1, x_2)$ of degree $m = 2k$ in three variables. The set

$$\mathbf{C}\mathcal{A} = \{(x_0 : x_1 : x_2) \in \mathbf{C}P^2 \mid F(x_0, x_1, x_2) = 0\}$$

is called *complex point set* of the curve. The complex conjugation *conj* of $\mathbf{C}P^2$ induces an antiholomorphic involution on $\mathbf{C}\mathcal{A}$ with fixed point set $\mathbf{R}\mathcal{A} = \mathbf{C}\mathcal{A} \cap \mathbf{R}P^2$, which is the set of real points of the curve. Given a real algebraic curve $\mathcal{A}$ with non-singular complex set point $\mathbf{C}\mathcal{A}$, using the fact that $\mathbf{C}P^2/conj$ is diffeomorphic to S^4 [**2**], one can associate to the triple $(\mathbf{C}P^2, conj, \mathbf{C}\mathcal{A})$ two pairs $(S^4, \mathfrak{A})$ where $\mathfrak{A}$ is a closed connected surface embedded in S^4 called *Arnold Surface*. These surfaces were introduced in [**3**]. The present paper is devoted to studying the topology of these pairs. The Arnold Surfaces are constructed as follows. The real part $\mathbf{R}\mathcal{A}$ is a union of disjoint circles embedded in $\mathbf{R}P^2$ which divide $\mathbf{R}P^2$ into two parts: $\mathbf{R}P^2_+ = \{x \in \mathbf{R}P^2 \mid F(x_0, x_1, x_2) \geq 0\}$ and $\mathbf{R}P^2_- = \{x \in \mathbf{R}P^2 \mid F(x_0, x_1, x_2) \leq 0\}$, which have common boundary $\mathbf{R}\mathcal{A}$. Gluing $\mathbf{R}P^2_+$, respectively $\mathbf{R}P^2_-$, to the quotient $\mathbf{C}\mathcal{A}/conj$ along their common boundary $\mathbf{R}\mathcal{A}$, we get two closed surfaces in S^4:

$$\mathfrak{A}_\pm = (\mathbf{C}\mathcal{A}/conj) \cup (\mathbf{R}P^2)_\pm$$

One of these surfaces is obviously non orientable. Changing the sign of F if necessary, we can always assume that the non-orientable part of $\mathbf{R}P^2$ lies entirely in $\mathbf{R}P^2_-$. Then $\mathfrak{A}_-$ is non-orientable. Note that these surfaces are not smooth along $\mathbf{R}\mathcal{A}$ but can be easily smoothed since $\mathbf{C}\mathcal{A}$ is nowhere tangent to $\mathbf{R}P^2$. In this paper we consider only the smoothed Arnold surfaces and we keep for them the same name "Arnold surfaces", since no confusion is possible. We shall say that an embedded surface $S \subset S^4$ is standard if it is a connected sum of standard tori and standard $\mathbf{R}P^2$. Moreover, when it does not lead to confusion, we shall call curve a real algebraic curve with non-singular complex point set.

Rokhlin's conjecture: *Let $\mathcal{A}$ be a real algebraic curve with non-empty real part $\mathbf{R}\mathcal{A}$.* [1] *Then both Arnold surfaces are standard.*

The proof of the Rokhlin's Conjecture for the surface $\mathfrak{A}_+$ associated to the maximal nest curves can be extracted from the paper [**1**] of S.Akbulut. (Recall that the maximal nest curve is a curve whose real part is , up to isotopy in $\mathbf{R}P^2$, k circles linearly ordered by inclusion.) In the paper [**8**], S.Finashin proved Rokhlin's Conjecture for L curves of even degree. (Recall that an L curve is a perturbation of a curve which splits into a union of $2k$ real lines in general position.)

Here is a brief description of the methods of our proof.

Let k be a positive integer. Consider an algebraic curve of degree $2k$ whose real part is described up to isotopy in $\mathbf{R}P^2$ as $(k-1)(2k-1)+1$ disjoint circles embedded in $\mathbf{R}P^2$ such that: $\frac{(k-1)(k-2)}{2}$ circles lie inside one circle and $\frac{k(k-3)}{2}$ circles lie outside this circle where the inside (resp, outside) of a circle design respectively the part of $\mathbf{R}P^2$ homeomorphic to a disc (resp, a Möbius band).

Such a curve was initially obtained in 1876 from Harnack's construction based on perturbations of singularities [**11**]. It can be as well obtained by the Patchworking procedure due to O.Viro. We recall the Patchworking procedure in the Preliminary Section. The construction of Harnack curve using the Patchworking

[1] In the case of empty real part, $\mathfrak{A}_-$ is trivially non-standard.

method was given by I.Itenberg in [**12**]. We recall these construction of the so-called Harnack curve in the Chapter 1.

Our proof starts in the Chapter 2. In the Chapter 3, we prove that Arnold surfaces for Harnack curves are standard. The proof makes use of a desingularization procedure.

Afterwards (sections 2 of Chapter 5, 3 of Chapter 5) we define a procedure of modification of the Harnack curve which allows to obtain (up to conj-equivariant isotopy) all real algebraic curves with non-empty real part. In the chapter 6 we prove that a curve $\mathcal{A}$ obtained as the result of such modification has standard Arnold surfaces if $\mathbf{R}\mathcal{A}$ is non-empty.

Acknowledgement

I am very grateful to Ilia Itenberg for posing the problem and for useful conversations during the course of this work.

Preliminaries-Patchworking Process

Introduction. The T-curves are constructed by a combinatorial procedure due to O.Viro [**18**]. They are naturally introduced via the procedure of patchworking polynomials.

Here we introduce some notations and definitions.

In what follows, $\mathbf{K}$ denotes either the real field $\mathbf{R}$ or the complex field $\mathbf{C}$; $\mathbf{Z}$ denotes the ring of integers, $\mathbf{N}$ the set of positive integers, $\mathbf{R}_+$ the set of positive reals.

A linear combination of products $\{x^i y^j \mid (i,j) \in \mathbf{Z}^2\}$ with coefficients from $\mathbf{K}$ is called an *L-polynomial* over $\mathbf{K}$ in two variables. L-polynomials over $\mathbf{K}$ in two variables form a ring isomorphic to the ring of regular function of the variety $(\mathbf{K}^*)^2$. The variety $(\mathbf{K}^*)^2$ is known as an algebraic torus over $\mathbf{K}$.
Let us recall some properties of $(\mathbf{K}^*)^2$.
Let l be the map $(\mathbf{K}^*)^2 \to \mathbf{R}^2$ defined by: $l(x,y) = (ln(|x|), ln(|y|))$. Let $U_{\mathbf{K}} = \{x \in \mathbf{K} \mid |x| = 1\}$ and a be the map $(\mathbf{K}^*)^2 \to U_{\mathbf{K}}^2$ defined by : $a(x,y) = (\frac{x}{|x|}, \frac{y}{|y|})$.
The map $la = (l,a) : (\mathbf{K}^*)^2 \to \mathbf{R}^2 \times U_{\mathbf{K}}^2$ is a diffeomorphism.
Being an abelian group $(\mathbf{K}^*)^2$ acts on itself by translations.
Let us recall some translations involved into this action. For $(i,j) \in \mathbf{R}^2$ and $t > 0$, denote by $qh_{(i,j),t} : (\mathbf{K}^*)^2 \to (\mathbf{K}^*)^2$ the translation defined by formula : $qh_{(i,j),t}(x,y) = (t^i x, t^j y)$. For $(x',y') \in U_{\mathbf{K}}^2$ denote by $S_{(x',y')}$ the translation $(\mathbf{K}^*)^2 \to (\mathbf{K}^*)^2$ defined by formula : $S_{(x',y')}(x,y) = (x'.x, y'.y)$.

Any usual real polynomial $f = f(x,y) = \sum_{(i,j)\in\mathbf{N}^2} a_{i,j} x^i y^j$ can be considered as a Laurent polynomial with coefficients from $\mathbf{K}$.

Let $f = f(x,y) = \sum_{(i,j)\in\mathbf{N}^2} a_{i,j} x^i y^j$ be a real polynomial in two variables. Denote by $\Delta(f)$ and call *Newton polyhedron* of f the convex hull of $\{w \in \mathbf{N}^2 \mid a_w \neq 0\}$.

By *polyhedron* we mean a convex bounded polyhedron in $\mathbf{R}^2$ with integer vertices having positive coordinates, in other words a Newton polyhedron of a real polynomial in two variables of finite degree.

The interior of a polyhedron Δ is denoted Δ^0.

A polyhedron is *non-degenerate* if its dimension is two, in other words its interior is non-empty.

A *subdivision* of a non-degenerate polyhedron Δ is a set of non-degenerate polyhedra $\Delta_1, ...\Delta_r$ such that $\Delta = \cup_i \Delta_i$, and any intersection $\Delta_i \cap \Delta_j$ is either

empty or is a common face of both Δ_i, Δ_j. We call *proper face* of a polyhedron Δ a face of codimension 1 and denote by $\mathcal{G}'(\Delta)$ the set of proper faces of a polyhedron Δ. The set of faces of a polyhedron Δ is denoted by $\mathcal{G}(\Delta)$.

We denote $C(\Delta)$ the vector subspace of $\mathbf{R}^2$ which corresponds to the minimal affine subspace containing Δ.

Toric varieties. To each polyhedron one can associate a variety called $\mathbf{K}$-toric variety. The following definition is deduced from ([**9**] p.166-168).

Definition 0.1. [**9**] *Let Δ be a polyhedron.*
Consider the finite set $A(\Delta) = \mathbf{Z}^2 \cap \Delta$ and choose an ordering of this set so that
$A(\Delta) = \{(i_1, j_1), ..., (i_n, j_n)\}$. Consider the subset

$$\mathbf{K}\Delta^0 = \{(x^{i_1}y^{j_1} : ... : x^{i_n}y^{j_n}) \mid (x, y) \in (\mathbf{K}^*)^2\}$$

of $\mathbf{K}P^{n-1}$.
The closure of $\mathbf{K}\Delta^0$ is called the $\mathbf{K}$-projective toric variety associated to Δ and the ordering of A.

Remark 0.2.
(1) Another choice of the ordering of A leads to an isomorphic $\mathbf{K}$-projective toric variety, so we shall use the term $\mathbf{K}$-projective toric variety associated to Δ whenever no confusion is possible.
(2) Moreover, let $B \subset \mathbf{Z}^2$ and $T : \mathbf{Z}^2 \to \mathbf{Z}^m$ be an affine integer injective transformation such that $T(A) = B$, the $\mathbf{K}$ projective variety associated to Δ and the set B is isomorphic to the $\mathbf{K}$-projective toric variety associated to Δ and an ordering set $A(\Delta)$.

Hence, denote by $\mathbf{K}\Delta$ the $\mathbf{K}$-projective toric variety associated to a polyhedron Δ.

Definition 0.3. [**9**]

Consider the action of the torus $(\mathbf{K}^*)^2$ given by formula $(x, y).(z_1 : ... : z_n) = (x^{i_1}y^{j_1}z_1 : ... : x^{i_n}y^{j_n}z_n)$, the variety $\mathbf{K}\Delta$ is the closure of the orbit of the point $(1 : ... : 1)$ under this action.

Let us recall some properties of toric varieties. We refer to [**18**],[**9**] for more details about toric varieties. The following Lemma is implicit in [**9**], Prop 1.9,p.171.

Lemma 0.4. [**9**] *Let Δ be a polyhedron. Let Γ be a face of Δ. Let $A(\Delta) = \mathbf{Z}^2 \cap \Delta$ and choose an ordering of this set so that $A(\Delta) = \{(i_1, j_1), ..., (i_n, j_n)\}$. Consider the subset $A(\Gamma) = \mathbf{Z}^2 \cap \Gamma$.*
Let $\partial : A(\Delta) \to 0, 1$ be the map such that $\partial(A(\Gamma)) = 1$ $\partial(A(\Delta) \backslash A(\Gamma)) = 0$. Consider the subset

$$\mathbf{K}'\Gamma^0 = \{(x^{i_1}y^{j_1}\partial((i_1, j_1)) : ... : (x^{i_n}y^{j_n}\partial((i_n, j_n)) \mid (x, y) \in (\mathbf{K}^*)^2\}$$

of $\mathbf{K}P^{n-1}$. The closure of $\mathbf{K}'\Gamma^0$ is a subvariety of $\mathbf{K}\Delta$ which corresponds to the closure of the orbit of the point $(\partial(i_1, j_1) : ... : \partial(i_n, j_n))$ under the action of the torus $(\mathbf{K}^*)^2$ given by formula $(x, y).(z_1 : ... : z_n) = (x^{i_1}y^{j_1}z_1 : ... : x^{i_n}y^{j_n}z_n)$.

Denote it by $\mathbf{K}'\Gamma$. The variety $\mathbf{K}'\Gamma$ is isomorphic to $\mathbf{K}\Gamma$ the $\mathbf{K}$-projective toric variety associated to Γ. In the cases when it does not lead to confusion we shall identify $\mathbf{K}'\Gamma$ with $\mathbf{K}\Gamma$ and $\mathbf{K}'\Gamma^0$ with $\mathbf{K}\Gamma^0$.

PROPOSITION 0.5. [**9**] *Let Δ a polyhedron. Let $\mathbf{K}\Delta$ be the $\mathbf{K}$-projective toric variety associated to Δ. Translations $qh_{(i,j),t}$ and $S_{(x',y')}$ of $(\mathbf{K}^*)^2$ define naturally translations $qh_{(i,j),t}$ and $S_{(x',y')}$ on $\mathbf{K}\Delta^0$ which can be extended to transformations on $\mathbf{K}\Delta$. In what follows, we keep the same notations for translations of $\mathbf{K}\Delta^0$ and their extensions to transformations of $\mathbf{K}\Delta$.*

proof:

On the assumptions of definition 0.1 of Chapter 0, let $\mathbf{K}\Delta$ be the $\mathbf{K}$-projective toric variety associated to Δ. Since $\mathbf{K}'\Gamma^0$ and their closure $\mathbf{K}'\Gamma$,with Γ a face of Δ, are contained in the affine parts of $\mathbf{K}P^{n-1}$, extensions of transformations $qh_{(i,j),t}$ and $S_{(x',y')}$ on varieties $\mathbf{K}'\Gamma$ give transformations on the affine parts of $\mathbf{K}P^{n-1}$ and thus define transformations of $\mathbf{K}\Delta$. Q.E.D

DEFINITION 0.6. [**9**] *Let Δ be a polyhedron. On the same assumptions as in Definition 0.1 of Chapter 0, let $\mathbf{K}\Delta$ be the closure of $\mathbf{K}\Delta^0$*

$$\mathbf{K}\Delta^0 = \{(x^{i_1}y^{j_1} : ... : x^{i_n}y^{j_n}) \mid (x,y) \in (\mathbf{K}^*)^2\}$$

subset of $\mathbf{K}P^{n-1}$. Let $\mathbf{R}_+\Delta$ be the closure (in the classic topology) of

$$\mathbf{R}_+\Delta^0 = \{(x^{i_1}y^{j_1} : ... : x^{i_n}y^{j_n}) \mid (x,y) \in (\mathbf{R}_+^*)^2\}$$

subset of $\mathbf{K}\Delta$. The subvariety $\mathbf{R}_+\Delta$ of $\mathbf{K}\Delta$ is homeomorphic,as stratified space,to the polyhedron Δ stratified by its faces. An explicit homeomorphism is given by the moment map $\mu : \mathbf{K}\Delta \to \Delta, \mu(x,y) = \frac{\sum_{(i,j)\in A(\Delta)} |x^iy^j|.(i,j)}{\sum_{(i,j)\in A(\Delta)} |x^iy^j|}$

proof:

It follows easily from the properties of the moment map μ([**5**]; see also [**9**], p.198 Theorem1.11) $\mu : \mathbf{K}\Delta \to \Delta$.Q.E.D

Consider the action of the torus $U^2_{\mathbf{K}}$ on $\mathbf{K}\Delta$ given by formula $(x,y).(z_1 : ... : z_n) = (x^{i_1}y^{j_1}z_1 : ... : x^{i_n}y^{j_n}z_n)$.

For a face Γ of Δ, denote by U_Γ the subgroup of $U^2_{\mathbf{K}}$ consisting of elements $(e^{i\pi k}, e^{i\pi l})$ with (k,l) a vector orthogonal to $C(\Gamma)$.

LEMMA 0.7. [**9**] *Let Δ be a polyhedron. The map $\rho : \mathbf{R}_+\Delta \times U^2_{\mathbf{K}} \to \mathbf{K}\Delta$ defined by formula $\rho((x,y),(x',y')) = S_{(x',y')}(x,y)$ is a proper surjection. The variety $\mathbf{K}\Delta$ is homeomorphic to the quotient space of $\mathbf{R}_+\Delta \times U^2_{\mathbf{K}}$ with respect to the partition into sets $x \times yU_\Gamma$, $x \in \mathbf{R}_+\Delta \cap \mathbf{K}\Gamma^0, y \in U^2_{\mathbf{K}}$.*

proof:
Let Δ be a polyhedron. Let $\mathbf{R}_+\Delta$ be the closure of

$$\mathbf{R}_+\Delta^0 = \{(x^{i_1}y^{j_1} : ... : x^{i_n}y^{j_n}) \mid (x,y) \in (\mathbf{R}_+^*)^2\}$$

subset of $\mathbf{K}\Delta$. Translations $S_{(x',y')}, (x',y') \in U^2_{\mathbf{K}}$ define an action of $U^2_{\mathbf{K}}$ in $\mathbf{K}\Delta$ such that intersection of $\mathbf{R}_+\Delta$ with each orbit under this action consists of one point. Furthermore, for x in the interior of Γ where Γ is a face of Δ, the stationary

subgroup of action of $U^2_{\mathbf{K}}$ consists of transformations $S_{(e^{i\pi k}, e^{i\pi l})}$ where (k,l) is a vector orthogonal to $C(\Gamma)$. Since $\mathbf{K}\Delta$ is locally compact and Hausdorff,it follows the Lemma. Q.E.D

Lemma 0.8. *The* $\mathbf{K}$*-projective toric variety associated to a non-degenerate triangle* Δ *is* $\mathbf{K}P^2$*. The* $\mathbf{K}$*-projective toric variety associated to a degenerate triangle* Δ *is* $\mathbf{K}P^1$*.*

proof:
It can be easily deduced from the Lemma 0.7 of Chapter 0 above. Let $\mathbf{K}=\mathbf{R}$. Place $\Delta \times U^2_R$ in $\mathbf{R}^2$ identifying $(x,y)(x',y') \in \Delta \times U^2_R$ with $S_{(x',y')}(x,y)$. The surface $\mathbf{R}\Delta$ can be obtained by an appropriate gluing of four copies of Δ.
Let $\mathbf{K}=\mathbf{C}$. Place $\Delta \times U^2_C$ in $\mathbf{R}^2$ identifying $(x,y)(x',y') \in \Delta \times U^2_C$ with $S_{(x',y')}(x,y)$. Let Δ be a non-degenerate triangle, it is not difficult to get the usual handlebody decomposition of $\mathbf{C}P^2$ as the union of three 4-balls. In the same way, let Δ be a degenerate triangle, it is not difficult to get $\mathbf{C}P^1$ as the union of two 2-discs.Q.E.D

Hypersurfaces of toric varieties. Let f be a real polynomial and $\Delta(f)$ its Newton polyhedron. Let Δ be a polyhedron such that $C(\Delta(f)) \subset C(\Delta)$. The equation $f=0$ defines in $\mathbf{K}\Delta^0$ an hypersurface $V_{\mathbf{K}\Delta^0}(f)$ of $\mathbf{K}\Delta^0$. Denote by $V_{\mathbf{K}\Delta}(f)$ its closure (in the Zarisky topology) in $\mathbf{K}\Delta$. (In the case $\mathbf{K}=\mathbf{C}$ the classic topology gives the same result, but in the case $\mathbf{K}=\mathbf{R}$ the usual closure may be a non-algebraic set)

Lemma 0.9. *([**18**], p.175) Let* f *a real polynomial and* $\Delta(f)$ *its Newton polyhedron. Let* Δ *a non-degenerate polyhedron (* $C(\Delta(f)) \subset C(\Delta)$*). For any vector* $(k,l) \in C(\Delta)$ *orthogonal to* $C(\Delta(f))$*, the hypersurface* $V_{\mathbf{K}_\Delta}(f)$ *is invariant under transformations* $S_{(e^{i\pi k}, e^{i\pi l})}$ *and* $qh_{(k,l),t}$ *of* $\mathbf{K}\Delta$ *.*

Let Δ be a polyhedron. An homeomorphism $\phi : \Delta \to \Delta$ is said *admissible* if it maps any face Γ of Δ to itself. (Thus, an admissible homeomorphism $\phi : \Delta \to \Delta$ extends to an homeomorphism $\tilde{\phi} : \Delta \times U^2_{\mathbf{K}} \to \Delta \times U^2_{\mathbf{K}}$ by equivariance).

Hence, the following definition is naturally introduced.

Definition 0.10. [**18**]

(1) *Let* f *be a real polynomial in two variables,* $\Delta(f)$ *its Newton polyhedron and* $\mathbf{K}\Delta(f)$ *the toric variety associated to* $\Delta(f)$*. Denote by* $h = \mu|^{-1}_{\mathbf{R}_+\Delta(f)}$ *the homeomorphism* $h : \Delta(f) \to \mathbf{R}_+\Delta(f)$*. Let* $\rho : \mathbf{R}_+\Delta(f) \times U^2_{\mathbf{K}} \to \mathbf{K}\Delta(f)$ *be the surjection defined by formula* $\rho((x,y),(x',y')) = S_{(x',y')}(x,y) = (x.x', y.y')$*.*

$$\Delta(f) \times U^2_{\mathbf{K}} \to^{h\times id} \mathbf{R}_+\Delta(f) \times U^2_{\mathbf{K}} \to^{\rho} \mathbf{K}\Delta(f)$$

A pair consisting of $(\Delta(f)) \times U^2_{\mathbf{K}}$ *and its subset* l *which is the pre-image of* $V_{\mathbf{K}\Delta(f)}$ *under* $\rho \circ (h \times id)$ *is a canonical* $\mathbf{K}$*-chart of* f*.*

(2) *Let* $(\Delta(f) \times U^2_{\mathbf{K}}, l)$ *be the canonical* $\mathbf{K}$*-chart of* f*, and* $\phi\ : \Delta(f) \times U^2_{\mathbf{K}} \to \Delta(f) \times U^2_{\mathbf{K}}$ *the extended homeomorphism of an admissible homeomorphism* $\phi : \Delta(f) \to \Delta(f)$*, the pair* $(\phi\ (\Delta(f) \times U^2_{\mathbf{K}}, \phi\ (l))$ *is a* $\mathbf{K}$*-chart of* f*.*

(3) *Let* $(\mathbf{R})^2_+$ *be set of positive integers* $\{(x,y) \in \mathbf{R}^2 \mid x \geq 0, y \geq 0\}$ *Consider the map:* $(\mathbf{R}^2)_+ \times U^2_{\mathbf{K}} \to \mathbf{C}^2 : ((x,y)(x',y')) \to S_{(x',y')}(x,y)$. *Call reduced* $\mathbf{K}$*-chart of* f *the image of a* $\mathbf{K}$*-chart of* f *under this map.*

Singular Hypersurfaces

Let $f = f(x,y) = \sum_{(i,j)\in\mathbf{N}^2} a_{i,j}x^iy^j$ be a real polynomial.

The set

$$SV_{(\mathbf{C}^*)^2}(f) = \{(x,y) \in (\mathbf{C}^*)^2 \mid f(x,y) = \frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) = 0\}$$

defines the set of *singular points* of the variety defined in $(\mathbf{C}^*)^2$ by the equation $f = 0$.

For a set $\Gamma \subset \mathbf{N}^2$ consider the polynomial $\sum_{(i,j)\in\Gamma} a_{i,j}x^iy^j$. It is called Γ-*truncation* of f and is denoted by $f^\Gamma(x,y)$.

The polynomial f is *completely non-degenerate* if for any face Γ (including $\Delta(f)$) of its Newton polyhedron $\Delta(f)$ the variety defined in $(\mathbf{C}^*)^2$ by the equation $f^\Gamma = 0$ is non-singular.

LEMMA 0.11. [**18**] *Completely non-degenerate real polynomials form a Zarisky open subset of the space of real polynomials with a given Newton polyhedron.*

1. Patchworking polynomials and T-curves

In this section, we shall explain the procedure of patchworking polynomials and define the concept of T-curve.

Patchworking polynomials. A reduced $\mathbf{R}$-chart of a non-degenerate polynomial $a(x,y) = a_{i_1,j_1}x^{i_1}y^{j_1} + a_{i_2,j_2}x^{i_2}y^{j_2} + a_{i_3,j_3}x^{i_3}y^{j_3}$ may be constructed from the following method.

Let $a = a(x,y) = a_{i_1,j_1}x^{i_1}y^{j_1} + a_{i_2,j_2}x^{i_2}y^{j_2} + a_{i_3,j_3}x^{i_3}y^{j_3}$ be a real polynomial with non-degenerate Newton polyhedron Δ. Assign to each vertex $a_{i,j}$ of Δ the sign

$$\epsilon_{i,j} = \frac{a_{i,j}}{|a_{i,j}|}$$

Consider the union of Δ and its symmetric copies $\Delta_x = s_x(\Delta)$ $\Delta_y = s_y(\Delta)$ $\Delta_{xy} = s(\Delta)$ where s_x, s_y are reflections with respect to the coordinate axes and $s = s_x \circ s_y$. For each vector (w_1, w_2) orthogonal to Δ with integer relatively prime coordinates glue the points (x,y) and $((-1)^{w_1}x, (-1)^{w_2}y)$ of the union $\Delta \cup \Delta_x \cup \Delta_y \cup \Delta_{xy}$.

Denote Δ_* the resulting space. Extend the distribution of signs to Δ_* so that $g^*(\epsilon_{i,j}x^iy^j) = \epsilon_{g(i,j)}x^iy^j$ for $g = s_x, s_y, s$. Let $\bigtriangledown$ be one of the four triangles $\Delta, \Delta_x, \Delta_y, \Delta_{xy}$. If $\bigtriangledown$ has vertices of different signs, consider the midline of $\bigtriangledown$ separating them. Denote by L the union of such midlines.

We shall say that *the pair* (Δ_*, L) *is obtained from a by combinatorial patchworking.*

LEMMA 1.1. [**18**] **"The smallest patch"**

Let $a(x,y) = a_{i_1,j_1}x^{i_1}y^{j_1} + a_{i_2,j_2}x^{i_2}y^{j_2} + a_{i_3,j_3}x^{i_3}y^{j_3}$ be a real polynomial with non-degenerate Newton polyhedron Δ. Let (Δ_*, L), obtained from $a(x,y)$ by combinatorial patchworking. The pair (Δ_*, L) is a reduced $\mathbf{R}$-chart of a.

It is easy to deduce the following characterization of the set of real points $\{(x,y) \in \mathbf{R}^2, a(x,y) = 0\}$ and $\{(x_0 : x_1 : x_2) \in \mathbf{R}P^2, A(x_0,x_1,x_2) = 0\}$ where A is the homogenization of a.

(1) Remove from Δ_* the sides of $\Delta \cup \Delta_x \cup \Delta_y \cup \Delta_{xy}$ which are not glued in the construction of Δ_*. It turns the polyhedron Δ_* to the polyhedron Δ' homeomorphic to $\mathbf{R}^2$ and the set L to a set L' such that the pair (Δ', v') is homeomorphic to $(\mathbf{R}^2, \{(x,y) \in \mathbf{R}^2, a(x,y) = 0\})$.

Glue by s the opposite sides of Δ_*. The resulting space $\bar{\Delta}_*$ is homeomorphic to the projective plane $\mathbf{R}P^2$. Denote $\bar{L}$ the image of L in $\bar{\Delta}_*$. Let $A(x_0,x_1,x_2)$ the homogenization of $a(x,y)$. It defines a curve $\mathcal{A}$. Then there exists an homeomorphism $(\bar{\Delta}_*, \bar{L}) \to (\mathbf{R}P^2, \mathbf{R}\mathcal{A})$.

One can generalize this description to more complicated polynomials .

Let $a_1, ..., a_r$ be completely non-degenerate real polynomials in two variables with Newton polyhedra $\Delta_1, ..., \Delta_r$ in such a way that $\Delta_1, ..., \Delta_r$ form a subdivision of a non-degenerate polyhedron Δ and $a_i^{\Delta_i \cap \Delta_j} = a_j^{\Delta_i \cap \Delta_j}$

Lemma 1.2. [**18**] **Patchworking polynomials**

Assume that the subdivision is regular, that is, there exists a convex non-negative function $\nu : \Delta \to \mathbf{R}$, satisfying the following conditions:

(1) *all the restrictions $\nu|\Delta_i$ are linear.*
(2) *if the restriction of ν to an open set is linear, then this set is contained in one of Δ_i*
(3) $\nu(\Delta \cap \mathbf{N}^2) \subset \mathbf{N}$.

Such a function is said convexifying the subdivision $\Delta_1, ..., \Delta_r$ of Δ.

There exists a unique polynomial a with $\Delta(a) = \Delta, a^{\Delta_i} = a_i$ for $i = 1, ..., r$. Let it be $a(x,y) = \sum_{(i,j)\in\mathbf{N}^2} a_{i,j}x^iy^j$.

Then introduce the one-parameter family of polynomials

$$b_t = b_t(x,y) = \sum_{(i,j)\in\mathbf{N}^2} a_{i,j}x^iy^jt^{\nu(i,j)}$$

We say that polynomials b_t are obtained by *Patchworking* the polynomials $a_1, ..., a_r$ by ν.

Definition 1.3. [**18**] **Patchworking Charts** *A pair $(\Delta \times U^2_{\mathbf{K}}, L)$ is said to be obtained by patchworking from $\mathbf{K}$-charts of polynomials $a_1, ..., a_r$ if $\Delta = \cup_{i=1}^r \Delta_i$ (where Δ_i denotes the Newton polyhedron of a_i) and one can choose $\mathbf{K}$-charts $(\Delta_i \times U^2_{\mathbf{K}}, l_i)$ of polynomials a_i such that $L = \cup_{i=1}^r (l_i)$.*

Theorem 1.4. [**18**] **Patchwork Theorem**[2] *Let $a_1, ..., a_r$ be completely non-degenerate polynomial in two variables with Newton polyhedra Δ_1,...,Δ_r. Assume*

[2]This theorem is a simplified version of the Patchworking Theorem for L-polynomials [**18**]

furthermore that Δ_1,...,Δ_r form a regular subdivision of a non-degenerate polyhedron Δ. Then there exists $t_0 > 0$ such that for any $t \in]0, t_0]$ the polynomial b_t is completely non-degenerate and its **K***-chart is obtained by patchworking* **K***-charts of the polynomials $a_1, ..., a_r$.*

Further, we say that such polynomials b_t are obtained by patchworking *process*.

We shall give the main ideas of the proof of the patchworking theorem in the section 0 of Chapter 0.

T-curves. Bringing together the Lemma 1.1 of Chapter 0 (”the smallest patch ”) and the patchworking theorem 1.4 of Chapter 0 we deduce a combinatorial method of construction of curves. These curves are called *T-curves*.

Let m be a positive integer. Let Δ be the triangle

$$\{(x, y) \in \mathbf{R}^2 \mid x \geq 0, y \geq 0, x + y \leq m\}$$

(Up to linear change of coordinates $(x_0 : x_1 : x_2)$ of $\mathbf{C}P^2$, the convex hull of Δ may be the Newton polyhedron of the affine polynomial $f(x, y) = F(1, x, y)$ associated to a homogeneous polynomial $F(x_0, x_1, x_2)$ of degree m.)

Let τ be a regular triangulation of Δ whose vertices have integer coordinates. Suppose that some distribution of signs χ at the vertices of the triangulation is given. Denote the sign $\pm$ at the vertex with coordinates (i, j) by $\epsilon_{(i,j)}$. Take the square Δ_* made of Δ and its symmetric copies $\Delta_x = s_x(\Delta), \Delta_y = s_y(\Delta), \Delta_{xy} = s_{xy}(\Delta)$ where $s_x, s_y, s = s_x \circ s_y$ are reflections with respect to the coordinate axes. The resulting space is homeomorphic to $\mathbf{R}^2$. Extend the triangulation τ of Δ to a symmetric triangulation τ_* of Δ_*. Extend the distribution of signs to a distribution at the vertices of Δ_* which verifies the modular properties: $g^*(\epsilon_{i,j})x^i y^j = \epsilon_{g(i,j)} x^i y^j$ for $g = s_x, s_y, s$.

If a triangle of Δ_* has vertices of different signs, consider a midline separating them. Denote by L the union of such midlines. Glue by s the opposite sides of Δ_*. The resulting space $\bar{\Delta}_*$ is homeomorphic to the projective plane $\mathbf{R}P^2$. Denote $\bar{L}$ the image of L in $\bar{\Delta}_*$. Call Δ_*, L, $\bar{\Delta}_*$,$\bar{L}$ obtained from Δ, τ, χ by combinatorial patchworking.

THEOREM 1.5. **Patchwork Theorem from the real view point and T-curves.**[**18**] *Define the one-parameter family of polynomials*

$$b_t = b_t(x, y) = \sum_{(i,j) vertices of \tau} \epsilon_{i,j} x^i y^j t^{\nu(i,j)}$$

where ν is a function convexifying the triangulation τ of Δ. Let, $\bar{\Delta}_*$,$\bar{L}$ obtained from Δ, τ, χ by combinatorial patchworking.

Denote by $B_t = B_t(x_0, x_1, x_2)$ the corresponding homogeneous polynomials:

$$B_t(x_0, x_1, x_2) = x_0^m b_t(x_1/x_0, x_2/x_0)$$

Then there exists $t_0 > 0$ such that for any $t \in]0, t_0]$ the equation $B_t(x_0, x_1, x_2) = 0$ defines in $\mathbf{R}P^2$ the set of real points of an algebraic curve C_t such that the pair $(\mathbf{R}P^2, \mathbf{R}C_t)$ is homeomorphic to the pair $(\bar{\Delta}_*, \bar{L})$.
Such a curve is called a T-curve .

proof :

It is an immediate consequence of the Lemma 1.2 of Chapter 0 and the theorem 1.4 of Chapter 0. Let Δ_*,L obtained from Δ, τ, χ by combinatorial patchworking. There exists t_0 such that for any $t \in]0, t_0]$ the pair (Δ_*, L) is a reduced $\mathbf{R}$-chart of b_t. Q.E.D

REMARK 1.6. (1) Remove the sides $x > 0, y > 0, x + y = m$ and its symmetric copies to Δ_*. It turns the polyhedron Δ_* to the polyhedron Δ' homeomorphic to $\mathbf{R}^2$ and the set L to a set L' such that for any $t \in]0, t_0]$ the pair (Δ', v') is homeomorphic to $(\mathbf{R}^2, \{(x, y) \in \mathbf{R}^2, b_t(x, y) = 0\})$.

(2) It is possible to recover in some cases whether the set of real points of the T-curve B divides the set of its complex points. Assume the triangulation τ of Δ sufficiently fine such that each triangle Δ_i in the triangulation τ is the Newton polyhedron of a polynomial a_i which defines a curve with orientable real set of points. Denote L_i the union of midlines in $(\Delta_i)_*$ homeomorphic to this set. If one can (resp, not) choose an orientation on each L_i compatible with an orientation of the union of $\cup_i(L_i)$ then the set of real points of the T-curve B divides (resp does not divide) the set of its complex points.

Main Ideas of the Proof of the Patchworking Theorem. In what follows, we shall give definitions and statements related to the patchworking method. Afterwards, we shall give a sketch of proof of the Patchwork Theorem.

Preliminaries

Let $a_1, ..., a_r$ be completely non-degenerate polynomials in two variables with Newton polyhedra Δ_1,...,Δ_r such that Δ_1,...,Δ_r form a regular subdivision of a non-degenerate triangle Δ. Let b_t be a polynomial obtained by patchworking process. Let B_t be the homogenization of b_t. It defines a real algebraic curve $\mathcal{C}_t$.

In this section, we explain the construction of b_t and give definitions we shall need in the next sections.

Recall that real polynomials in two variables belong to the ring of L-polynomials, which is isomorphic to the ring of regular function of the variety $(\mathbf{K}^*)^2$.

As already introduced, set la the diffeomorphism

$$la = (l, a) : (\mathbf{K}^*)^2 \to \mathbf{R}^2 \times U_{\mathbf{K}}^2$$

Call *ϵ-tubular neighborhood* N of a smooth submanifold M of $(\mathbf{K}^*)^2$ the normal tubular neighborhood of the smooth submanifold $la(M)$ of $\mathbf{R}^2 \times U_{\mathbf{K}}^2$ whose fibers lie in fibers $\mathbf{R} \times t \times U_{\mathbf{K}} \times s$, consist of segments of geodesics which are orthogonal to intersection of $la(M)$ with these $\mathbf{R} \times t \times U_{\mathbf{K}} \times s$, and are contained in balls of radius ϵ centered in the points of intersection of $la(M)$ with these $\mathbf{R} \times t \times U_{\mathbf{K}} \times s$. The intersection of such tubular neighborhood of M with the fiber $\mathbf{R} \times t \times U_{\mathbf{K}} \times s$ is a normal tubular neighborhood of $M \cap \mathbf{R} \times t \times U_{\mathbf{K}} \times s$ in $R \times t \times U_{\mathbf{K}} \times s$. (Obviously, such definition of ϵ *tubular neighborhood* of M in $(\mathbf{K}^*)^2$ requires $la(M)$ transversal to $R \times t \times U_{\mathbf{K}} \times s$.)

The following statement describes cases for which such tubular neighborhood exists.

We refer to ([**13**] p.59) and also to([**18**] p.175, p178) for its proof.

LEMMA 1.7. [**13**] *Let f be a polynomial and $\Delta(f)$ its Newton polyhedron. Let Γ be a face of $\Delta(f)$ such that f^Γ is completely non- degenerate. Let $\rho : \mathbf{R}_+\Delta(f) \times U^2_{\mathbf{K}} \to \mathbf{K}\Delta(f)$ the natural surjection . Then the hypersurface $\rho^{-1}(V_{\mathbf{K}\Delta(f)}(f))$ is transversal to $\mathbf{R}_+\Gamma \times U^2_{\mathbf{K}}$.*

Let M be a smooth manifold of $(\mathbf{K}^*)^2$ and Δ be a non-degenerate polyhedron. Denote $\rho : \mathbf{R}_+\Delta \times U^2_{\mathbf{K}} \to \mathbf{K}\Delta$ the natural surjection.

Using the moment map $\mu : \mathbf{K}\Delta \to \Delta$ which maps the torus $(\mathbf{K}^*)^2 \approx \mathbf{K}\Delta^0$ onto the interior Δ^0 of Δ, one can identify points of $\mathbf{R}_+\Delta^0$ with points of Δ^0. In such a way, using this identification, we call ϵ-tubular neighborhood of M defined from p in Δ^0, the ϵ-tubular neighborhood of M in $\rho(D(p,\epsilon) \times U^2_{\mathbf{K}}) \subset (\mathbf{K}^*)^2$ where $D(p,\epsilon)$ is an open (euclidian) disc around p of radius ϵ inside Δ^0. Thereby, given a triangulation of Δ, one can choose $D(p,\epsilon)$ sufficiently small such that $D(p,\epsilon)$ intersects only one proper Γ face of the triangulation.

DEFINITION 1.8. [**18**] *Let M a smooth manifold of $(\mathbf{K}^*)^2$. Let Δ be a non-degenerate polyhedron and $\rho : \mathbf{R}_+\Delta \times U^2_{\mathbf{K}} \to \mathbf{K}\Delta$ be the natural surjection. Given $\epsilon > 0$. We call ϵ-tubular neighborhood of $M \subset (\mathbf{K}^*)^2$ defined from a point p in $\mathbf{R}_+\Delta^0$, the ϵ-tubular neighborhood of M in $U(p) = \rho(D(p,\epsilon) \times U^2_{\mathbf{K}}) \subset (\mathbf{K}^*)^2$ where $D(p,\epsilon)$ is an open (euclidian) disc around p of radius ϵ contained in $\mathbf{R}_+\Delta^0$.*

Let $D(p,\epsilon) \in \mathbf{R}_+T^0_m$ be an open (euclidian) disc around p of radius ϵ such that $\mu(D(p,\epsilon))$ intersects only the face Γ of the triangulation of T_m and contains Γ. We call ϵ-tubular neighborhood of M defined from points of Γ^0, the ϵ-tubular neighborhood of M in $U(p) = \rho(D(p,\epsilon) \times U^2_{\mathbf{K}}) \subset (\mathbf{K}^*)^2$. We call $U(p) = \rho(D(p,\epsilon) \times U^2_{\mathbf{K}}) \subset (\mathbf{K}^*)^2 \subset (\mathbf{K}^*)^2$ the ϵ-neighborhood of M defined from Γ^0.

DEFINITION 1.9. [**18**] *Let the norm in vector spaces of L-polynomials be:*

$$||\sum a_w x^w|| = max\{|a_w||w \in \mathbf{Z}^2\}$$

Let a a L-polynomial over $\mathbf{K}$ and U a subset of $(\mathbf{K}^*)^2$ we say that in U the truncation a^Γ is ϵ-sufficient for a if for any L-polynomial b over K such that $\Delta(b) \subset \Delta(a), b^\Gamma = a^\Gamma, ||b - b^\Gamma|| \leq ||a - a^\Gamma||$ the following conditions are verified:

1. *$U \cap SV_{(\mathbf{K}^*)^2}(b) = \emptyset$*
2. *the set $la(U \cap V_{(\mathbf{K}^*)^2}(b))$ lies in a tubular ϵ-neighborhood of $la(V_{(\mathbf{K}^*)^2}(a^\Gamma) \backslash SV_{(\mathbf{K}^*)^2}(a^\Gamma))$*
3. *$la(U \cap V_{(\mathbf{K}^*)^2}(b))$ can be extended to the image of a smooth section of the tubular fibration $N \to la(V_{(\mathbf{K}^*)^2}(a^\Gamma) \backslash SV_{(\mathbf{K}^*)^2}(a^\Gamma))$*

Let Δ a polyhedron. Denote by $C_\Delta(\Gamma)$ the cone $C_\Delta(\Gamma) = \cup_{r \in \mathbf{R}_+} r.(\Delta - y)$, where y is a point of $\Gamma \backslash \partial\Gamma$. It contains the minimal affine subspace $C(\Gamma)$ of $\mathbf{R}^2$ containing Γ, $C(\Gamma) \subset C_\Delta(\Gamma)$. Denote by $DC^-_\Delta(\Gamma)$ the set $\{x \in \mathbf{R}^2, a \in C_\Delta(\Gamma) a.x \leq 0\}$. For $A \subset \mathbf{R}^2$ and $\rho > 0$, set $\mathcal{N}_\rho(A) = \{x \in \mathbf{R}^2 \backslash dist(x, A) < \rho\}$. Let $\phi : \mathcal{G}(\Delta) \to \mathbf{R}$ be a positive function, denote by $DC_{\Delta,\phi}(\Gamma)$ the set $\mathcal{N}_{\phi(\Gamma)}(DC^-_\Delta(\Gamma)) \backslash \cup_{\Sigma \in \mathcal{G}(\Delta), \Gamma \in \mathcal{G}(\Sigma)}$ $\mathcal{N}_{\phi(\Sigma)}(DC^-_\Delta(\Sigma))$.

DEFINITION 1.10. [**18**] *Let f be a Laurent polynomial over $\mathbf{K}$ and Δ its Newton polyhedron. A positive function $\phi : \mathcal{G}(\Delta) \to \mathbf{R}$ describes domain of ϵ-sufficiency*

for f if for any proper face $\Gamma \in \Gamma(\Delta)$, for which truncation is completely non-degenerate and the hypersurface $la(V_{\mathbf{KR}^2}(f^\Gamma))$ has an ϵ-tubular neighborhood, the truncation f^Γ is ϵ-sufficient for f in some neighborhood of $l^{-1}(DC_{\Delta,\phi}(\Gamma))$.

Main Ideas of the proof of the Patchworking Theorem

We shall just give the main ideas of a proof of the Patchworking theorem. We refer the reader to [**18**] for the complete proof.

First notice that from the Lemma 1.7 of Chapter 0 since polynomials a_1,...,a_r are completely non-degenerate one can consider ϵ-tubular neighborhood around points of any hypersurface $a_i = 0$ in $\mathbf{K}\Delta_i$ which belong to $\mathbf{K}\Gamma$ with $\Gamma \in \mathcal{G}'(\Delta_i)$.

Besides, recall that to each non-degenerate polyhedron Δ one can associate projective toric variety $\mathbf{K}\Delta$. The variety $\mathbf{K}\Delta$ can be seen as the completion of $(\mathbf{K}^*)^2$ in such a way that the toric varieties associated to the proper faces of Δ cover $\mathbf{K}\Delta \backslash (\mathbf{K}^*)^2$.

Recall the following statement extracted from [**18**].

Lemma 0 [**18**]

For any polynomial f and $\epsilon > 0$ there exists a function $\phi : \mathcal{G}(\Delta(f)) \to \mathbf{R}$ describing domain of ϵ-sufficiency for f.

Denote $\mathcal{G} = \cup_{i=1}^r \mathcal{G}(\Delta_i)$. Define $b(x, y, t) = b_t(x, y)$ the polynomial in the three variables (x, y, t). Denote Δ" the Newton polyhedron of b. Δ" is the convex hull of the graph of ν. For $\Gamma \in \mathcal{G}$ denote by Γ" the face of Δ" which is the graph of $\nu|_\Gamma$. For $t > 0$, let j_t be the embedding $\mathbf{R}^2 \to \mathbf{R}^3$ given by formula $j_t(x, y) = (x, y, lnt)$. Let $\psi : \mathcal{G} \to \mathbf{R}$ be a positive function. For $\Gamma \in \mathcal{G}$ denote by $\mathcal{E}_{t,\psi}$ the subset :

$$\mathcal{N}_{\psi(\Gamma)} j_t^{-1}(DC^-_{\Delta"}(\Gamma"))\backslash \cup_{\Sigma\in\mathcal{G}(\Delta),\Gamma\in\mathcal{G}(\Sigma)} \mathcal{N}_{\phi(\Sigma)} j_t^{-1}(DC^-_{\Delta"}(\Sigma"))$$

The following Lemma is extracted from [**18**].

Lemma 1 [**18**] *Let a_1,..a_r be completely non-degenerate polynomials. For any $\epsilon > 0$, there exists a function $\psi : \mathcal{G} \to \mathbf{R}$ such that for any $t \in]0, t_0]$ and any face $\Gamma \in \mathcal{G}$, there exists a neighborhood $\mathcal{E}_{t,\psi}(\Gamma)$ of Γ such that b_t^Γ is ϵ-sufficient for b_t in $l^{-1}\mathcal{E}_{t,\psi}(\Gamma)$.*

sketch of proof:
It follows from the existence of a positive function $\psi : \mathcal{G} \to \mathbf{R}$ such that for each proper face Γ" of Δ", $b^{\Gamma"}$ is ϵ-sufficient for b in a neighborhood which contains $\mathcal{E}_{t,\psi}(\Gamma)$ for any t sufficiently small. Q.E.D.

Let b_t be a polynomial obtained by patchworking the polynomials a_1,...a_r. The Newton polyhedron $\Delta(b_t)$ of the polynomial b_t is the polyhedron Δ. The following lemma follows immediately from Lemma 0.

Lemma 2[**18**]

For any polynomial b_t obtained by patchworking the polynomials a_1,...a_r and $\epsilon > 0$ there exists a function $\phi : \mathcal{G}(\Delta) \to \mathbf{R}$ describing domain of ϵ-sufficiency for b_t.

Besides (see [**18**]), for a well chosen function $\phi : \mathcal{G}(\Delta) \to \mathbf{R}$, the ϵ-sufficiency of b_t^Γ for b_t in $l^{-1}\mathcal{E}_{t,\psi}(\Gamma)$ for $\Gamma \in \mathcal{G}(\Delta)\backslash\Delta$ implies ϵ-sufficiency for b_t in some neighborhood of $l^{-1}(DC_{\Delta,\phi}(\Gamma))$ and of $l^{-1}(DC_{\Delta,\phi}(\Delta))$ in such a way that for any

$t \in]0, t_0]$ a **K**-chart of the polynomial b_t is obtained by patchworking **K**-chart of the polynomials $a_1, ..., a_r$. Hence, it follows the Patchworking Theorem.

Furthermore, one can make the following remark.
Since polynomials $a_{i, i\in\{1,...,r\}}$ and $b_t^{\Delta_i}$ for t sufficiently small are completely non-degenerate, from the Lemma 1 and the existence of a well chosen function $\phi : \mathcal{G}(\Delta) \to \mathbf{R}$ as above, the following approximations of $V_{\mathbf{K}\Delta}(b_t)$ can be easily deduced.

Denote the gradient of the restriction of ν on Γ by $\nabla(\nu|_\Gamma)$. The truncation b_t^Γ equals $a^\Gamma \circ qh_{\nabla(\nu|_\Gamma),t}$ where a is the unique polynomial with $\Delta(a) = \Delta, a^{\Delta_i} = a_i$ for $i = 1, ..., r$. From the Lemma 1, we get that for $t \in]0, t_0]$ the space $\mathbf{KR}^2$ is covered by regions in which $V_{\mathbf{KR}^2}(b_t)$ is approximated by $qh^{-1}_{\nabla(\nu|_{\Delta_i}),t}(V_{\mathbf{KR}^2}(a_i))$. Extending translations $qh_{\nabla(\nu|_\Gamma),t}$ to $\Gamma \in \mathcal{G}(\Delta_i)$ and $S_{(x',y')}(x, y)$ to the whole space $\mathbf{K}\Delta$, it follows the description of $V_{\mathbf{K}\Delta}(b_t)$.

2. Metric on $\mathbf{C}P^2$ and $\mathbf{R}P^2$

The projective space $\mathbf{C}P^2$ as any differentiable manifold admits a Riemannian Metric. In what follows, we shall present the Riemannian Metric of $\mathbf{C}P^2$ usually called Fubini-Study metric.

Local Euclidian metric. The topological space $\mathbf{C}P^2$, as complex manifold looks like locally as the 2-dimensional affine space $\mathbf{C}^2$. In other words, $\mathbf{C}P^2$ admits a covering by open set $\mathbf{C}U_i = \{(z_0 : z_1 : z_2) \in \mathbf{C}P^2 | z_i \neq 0\}$, $i \in \{0, 1, 2\}$ provided with charts $\phi_i : \mathbf{C}U_i \to \mathbf{C}^2$, $i \in \{0, 1, 2\}$ ($\phi_1(z_0 : z_1 : z_2) = (\frac{z_0}{z_1}, \frac{z_2}{z_1})$; ϕ_0 and ϕ_2 are symmetrically defined), where all the $\phi_i \circ \phi_j^{-1}$ are complex analytic.

With their help, one can use the cartesian coordinates in $\mathbf{C}^2$ as local coordinates in $\mathbf{C}U_i \subset \mathbf{C}P^2$ and carry out complex analysis on $\mathbf{C}P^2$.

In such a way, we may locally define 4-ball of $\mathbf{C}P^2$ in an open $\mathbf{C}U_i$ as $\phi_i^{-1}(B^4) \supset \mathbf{C}U_i$ where B^4 is a usual 4-ball of $\mathbf{C}^2$. Passing from complex to real set of points, any 2-disc of $\mathbf{R}P^2 \subset \mathbf{C}P^2$ is defined in an open $\mathbf{R}U_i \subset \mathbf{C}U_i$, as $\phi_i^{-1}(D^2) \supset \mathbf{R}U_i$ where D^2 is a usual 2-disc of $\mathbf{R}^2$.

Riemannian metric. One can also provide $\mathbf{C}P^2$ with another structure by giving a euclidian metric on the tangent space of each point $p \in \mathbf{C}P^2$ that is by introducing a Riemannian metric on the complex projective space $\mathbf{C}P^2$. This metric on $\mathbf{C}P^2$ is called the Fubini-Study metric.

Let $S^5 = \{(z_0, z_1, z_2) \in \mathbf{C}^3 | z_0.\bar{z_0} + z_1.\bar{z_1} + z_2.\bar{z_2} = 1\}$ and $\pi_\mathbf{C} : S^5 \to \mathbf{C}P^2$ be the natural projection. For any point $p \in \mathbf{C}P^2$ choose a representative $(p_0 : p_1 : p_2)$ with $p_0^2 + p_1^2 + p_2^2 = 1$, $\pi_\mathbf{C}^{-1}(p)$ is the circle $\{p' | p' = \lambda.(p_0, p_1, p_2)\ with\ \lambda \in \mathbf{C}, |\lambda| = 1\}$.

Given two points $p = (p_0 : p_1 : p_2) \in \mathbf{C}P^2$, $q = (q_0 : q_1 : q_2) \in \mathbf{C}P^2$,($p_0^2 + p_1^2 + p_2^2 = 1$, $q_0^2 + q_1^2 + q_2^2 = 1$), we define the distance δ between p and q in the Fubini-Study metric as the length of the geodesic between the circles $\pi_\mathbf{C}^{-1}(p)$ $\pi_\mathbf{C}^{-1}(q)$ of S^5.

It is easy to verify that the Fubini-Study metric is invariant under the action of complex conjugation. Besides, when consider the restriction of S^5 to its real set

of points S^2, one define the metric on $\mathbf{R}P^2$.
Denote $\pi_{\mathbf{R}} : S^2 \to \mathbf{R}P^2$ the natural projection.

Given two points $p = (p_0 : p_1 : p_2) \in \mathbf{R}P^2$, $q = (q_0 : q_1 : q_2) \in \mathbf{R}P^2$, $(p_0^2 + p_1^2 + p_2^2 = 1,\ q_0^2 + q_1^2 + q_2^2 = 1)$, $\pi_{\mathbf{R}}^{-1}(p) = \{(p_0, p_1, p_2), (-p_0, -p_1, -p_2)\}$ $\pi_{\mathbf{R}}^{-1}(q) = \{(q_0, q_1, q_2), (-q_0, -q_1, -q_2)\}$, noticing that the antipodal mapping $A : S^2 \to S^2$ is an isometry, we define the distance δ between p and q in the Fubini-Study metric as the minimal arc length of the arcs of S^2 between $(p_0, p_1, p_2) \in \pi_{\mathbf{R}}^{-1}(p)$, and $\pi_{\mathbf{R}}^{-1}(q) = \{(q_0, q_1, q_2), (-q_0, -q_1, -q_2)\}$.
Given two real points of $S^2 \subset S^5$, it is easy to see (since geodesic minimizes arc length between two of its points) that any point of the geodesic from these two points is real.
In such a way, in the Fubini-Study metric of $\mathbf{C}P^2$, we define any 4-ball of $\mathbf{C}P^2$ by $\pi_{\mathbf{C}}(\mathcal{E})$ where $\mathcal{E}$ is an an ellipsoid of S^5; and any 2-disc of $\mathbf{R}P^2 \subset \mathbf{C}P^2$ by $\pi_{\mathbf{C}}(E) = \pi_{\mathbf{R}}(E)$ where E is an ellipse of $S^2 \subset S^5$.

Part 2

Arnold Surfaces of Harnack Curves

The maximal number [**11**] of connected components of the real point set of curves of degree m is $\frac{(m-1)(m-2)}{2}+1$. Curves with this maximal number are called M-curves. In this section, we study the construction of some M-curves called Harnack curves and prove that Arnold surfaces of the so-called Harnack curves of even degree are standard.

CHAPTER 1

Combinatorial patchworking construction for Harnack curves

Preliminaries Recall that the pair $(\mathbf{R}P^2, \mathbf{R}\mathcal{A})$ where $\mathcal{A}$ is a non-singular real plane curve is determined up to homeomorphism by the real components of the curve and their relative location.

In the case of even degree curve $\mathcal{A}$, each connected component of the real point set $\mathbf{R}\mathcal{A}$ is situated in $\mathbf{R}P^2$ as the boundary of an embedded disc and is called an *oval*.

In the case of odd degree curve $\mathcal{A}$, the real point set $\mathbf{R}\mathcal{A}$ has besides oval one connected component situated in $\mathbf{R}P^2$ as an embedded projective line. It is called the *one side component* of $\mathbf{R}\mathcal{A}$. An oval divides $\mathbf{R}P^2$ into two components. The orientable component (i.e the component homeomorphic to a disc) is called the inside of the oval. The non-orientable component (i.e the component homeomorphic to a Möbius strip) is called the outside of the oval.

Given $\mathcal{A}$ a non-singular real plane curve and $F(x_0, x_1, x_2)$ its polynomial, denote by $\mathbf{R}P^2_+$ the subset of $\mathbf{R}P^2$ $\{x \in \mathbf{R}P^2 \ |F(x_0, x_1, x_2) \geq 0\}$. An oval is said *outer* (resp *inner*) if it bounds a component of $\mathbf{R}P^2_+$ from the outside (resp, the inside). In the case of an even degree curve $\mathcal{A}$, we can always assume (changing the sign of the polynomial $F(x_0, x_1, x_2)$ giving $\mathcal{A}$ if necessary) that $\mathbf{R}P^2_+ = \{x \in \mathbf{R}P^2 \ |F(x_0, x_1, x_2) \geq 0\}$ is the orientable component of $\mathbf{R}P^2$. In such away, in case of an even degree curve, since ovals lying in an even number of consecutive ovals are outer while the ovals lying in an odd number of consecutive ovals are inner; one also calls *even* ovals the outer ovals and *odd* ovals the inner ovals.

In the case of an odd degree curve $\mathcal{A}$, the definition of outer and inner oval apply only to oval which does not intersect the line at infinity. Call *zero oval* an oval of a curve of odd degree intersecting the line at infinity.

The pair $(\mathbf{R}P^2, \mathbf{R}\mathcal{A})$ where$\mathcal{A}$ is a non singular curve of degree m is defined by the scheme of disposition of the real component $\mathbf{R}\mathcal{A}$. This scheme is called the *real scheme of the curve* $\mathcal{A}$. In what follows, we use the following usual system of notations for real scheme (see [**18**]). A single oval is denoted by $\langle 1\rangle$. The one side component is denoted by $\langle J\rangle$. The empty real scheme is denoted by $\langle 0\rangle$. If one set of ovals is denoted by $\langle A\rangle$, then the set of ovals obtained by addition of an oval which contained $\langle A\rangle$ in its inside component is denoted by $\langle 1\langle A\rangle\rangle$. If the real scheme of a curve consists of two disjoint sets of ovals denoted by $\langle A\rangle$ and$\langle B\rangle$ in such a way that no oval of one set contains an oval of the other set in its inside component, then the real scheme of this curve is denoted by : $\langle A \sqcup B\rangle$. If one set

of ovals is denoted by A, then the set $A \cup ... \cup A$ where A occurs n times, is denoted by $\langle n \times A \rangle$; a set $\langle n \times 1 \rangle$ is denoted by $\langle n \rangle$.

Furthermore, if $\mathcal{A}$ is a curve of which the set of real points $\mathbf{R}\mathcal{A}$ divides the set of complex points $\mathbf{C}\mathcal{A}$ in two connected components (which induces two opposite orientations on $\mathbf{R}\mathcal{A}$), then the curve $\mathcal{A}$ is said of type I. Otherwise, $\mathcal{A}$ is said of type II. Hence, the real scheme of a curve $\mathcal{A}$ of degree m is of type I (resp, of type II), if any curve of degree m having this real scheme is of type I(resp, of type II). Otherwise (i.e if there exists both curves of type I and curves of type II with the given real scheme), we say that the real scheme is of *indeterminate type*.

Thus, isotopy of $\mathbf{C}P^2$ which connects complex points of curves and commutes with the complex conjugation of $\mathbf{C}P^2$, called *conj-equivariant* isotopy of $\mathbf{C}P^2$, provides a convenient way to classify pairs $(\mathbf{C}P^2, \mathbf{C}\mathcal{A})$. We can already notice that since M-curves are curves of type I, their real scheme is sufficient to classify their complex set of points in $\mathbf{C}P^2$ up to conj-equivariant isotopy.

1. Constructing Harnack curves with Harnack's initial method

In 1876, Harnack proposed a method for constructing M-curves. Recall briefly Harnack's initial construction [**11**] of Harnack's M-curves. (The detailed method can be found in [**10**])

Start with a line as M-curve of degree 1. Then, consider a line L which intersects it in one point. Assume an M-curve of degree m $\mathcal{A}_m$ has been constructed at the step m of the construction. At the step $m+1$ of the construction, the M-curve of degree $m+1$ is obtained from classical deformation (see [**18**] Classical Small Perturbation Theorem for the definition of classical deformation) of $\mathcal{A}_m \cup L$. The resulting M-curve of degree $m+1$ intersects the line L in $m+1$ real points.

Passing from curves to polynomials giving these curves, an M-curve $\tilde{X}_{m+1} = 0$ is constructed from an M-curve $\tilde{X}_m = 0$ of degree m by the formula $\tilde{X}_{m+1} = x_0.\tilde{X}_m + t.C_{m+1}$ where x_0 is a line, $C_{m+1} = 0$ is $m+1$ parallels lines which intersect $x_0 = 0$ in $m+1$ points. It is essential in the construction that the curves $\tilde{X}_m = 0$, $x_0 = 0$ and $C_{m+1} = 0$ do not have common points.

Thus, one can choose projective coordinates $(x_0 : x_1 : x_2)$ of $\mathbf{C}P^2$, in such a way that $\tilde{X}_m = 0$ is an homogeneous polynomial of degree m in the variables x_0, x_1,x_2 and $C_{m+1} = 0$ is an homogeneous polynomial of degree $m+1$ in the variables x_2,x_1. For sufficiently small $t > 0$, $\tilde{X}_{m+1} = 0$ is an Harnack curve of degree $m+1$.

In particular, this method provides curves with real scheme:
for even $m = 2k$

$$\Big\langle 1\Big\langle \frac{(k-1)(k-2)}{2} \Big\rangle \sqcup \frac{3k(k-1)}{2} \Big\rangle \tag{1}$$

for odd $m = 2k+1$

$$\Big\langle J \sqcup k(2k-1) \Big\rangle \tag{2}$$

Let us denote $\mathcal{H}_m$ and call *Harnack curve of degree m*, any curve of degree m with the real scheme above. Besides, we shall call *Harnack polynomial* of degree m any polynomial giving the Harnack curve of degree m.

We shall call curve of type $\mathcal{H}$ a curve $\mathcal{H}_m$ which intersects the line at infinity of $\mathbf{C}P^2$ in m real distinct points. Without loss of generality, one can always assume that the line L is the line at infinity. Therefore, curves $\mathcal{H}_m$ constructed by the Harnack's method are curves of type $\mathcal{H}$.

2. Patchworking construction for Harnack curves

In what follows, we shall give a construction of Harnack curves provided by the patchworking construction of curves .

Recall briefly the patchworking construction procedure of curves due to Viro [**18**]. (We refer the reader to [**18**] and also to the Preliminary section for details.)

Initial data

(1) Let m be a positive integer. Let T_m be the triangle

$$\{(x,y) \in \mathbf{R}^2 \mid x \geq 0, y \geq 0, x+y \leq m\}$$

(Up to linear change of coordinates $(x_0 : x_1 : x_2)$ of $\mathbf{C}P^2$, the convex hull of T_m may be the Newton polyhedron of the affine polynomial $f(x,y) = F(1,x,y)$ associated to a homogeneous polynomial $F(x_0,x_1,x_2)$ of degree m.)

(2) Let τ be a triangulation of T_m whose vertices have integer coordinates. Call regular triangulation of T_m a triangulation of T_m such that there exists with a convexifying function $\nu : T_m \to \mathbf{R}$ (that is a piecewise-linear function $\nu : T_m \to \mathbf{R}$ which is linear on each triangle of the triangulation τ and not linear on the union of two triangles.) Assume τ regular.

(3) Suppose that some distribution of signs χ at the vertices of the triangulation is given. Denote the sign $\pm$ at the vertex with coordinates (i,j) by $\epsilon_{(i,j)}$.

Combinatorial procedure

Take the square T_m^* made of T_m and its symmetric copies $T_m^x = s_x(T_m), T_m^y = s_y(T_m), T_m^{xy} = s_{xy}(T_m)$ where $s_x, s_y, s = s_x \circ s_y$ are reflections with respect to the coordinate axes. The resulting space is homeomorphic to $\mathbf{R}^2$. Extend the triangulation τ of T_m to a symmetric triangulationsτ_* of T_m^*. Extend the distribution of signs to a distribution at the vertices of T_m^* which verifies the modular properties: $g^*(\epsilon_{i,j})x^iy^j = \epsilon_{g(i,j)}x^iy^j$ for $g = s_x, s_y, s$.

If a triangle of T_m^* has vertices of different signs, consider a midline separating them. Denote by L the union of such midlines. Glue by s the opposite sides of T_m^*. The resulting space $\bar{T}_m^*$ is homeomorphic to the projective plane $\mathbf{R}P^2$. Denote $\bar{L}$ the image of L in $\bar{T}_m^*$. We shall say that T_m^*, L, $\bar{T}_m^*$,$\bar{L}$ are obtained from T_m, τ, χ by combinatorial patchworking.

THEOREM 2.1. **Polynomial Patchworking** *Define the one-parameter family of polynomials*

$$b_t = b_t(x,y) = \sum_{(i,j) vertices of \tau} \epsilon_{i,j} x^i y^j t^{\nu(i,j)}$$

where ν is a function convexifying the triangulation τ of T_m. Let $\bar{T}_m^*$,$\bar{L}$ obtained from T_m, τ, χ by combinatorial patchworking.

Denote by $B_t = B_t(x_0, x_1, x_2)$ *the corresponding homogeneous polynomials:*

$$B_t(x_0, x_1, x_2) = x_0^m b_t(x_1/x_0, x_2/x_0)$$

Then there exists $t_0 > 0$ *such that for any* $t \in]0, t_0]$ *the equation* $B_t(x_0, x_1, x_2) = 0$ *defines in* $\mathbf{R}P^2$ *the set of real points of an algebraic curve* C_t *such that the pair* $(\mathbf{R}P^2, \mathbf{R}C_t)$ *is homeomorphic to the pair* $(\bar{T}_m^*, \bar{L})$*. Such a curve is called a* T*-curve* .

Let m be a positive integer.
Let T_m be the triangle

$$\{(x,y) \in \mathbf{R}^2 \mid x \geq 0, y \geq 0, x+y \leq m\}$$

Define the regular triangulation τ of T_m as follows:

(1) All integer points of T_m are vertices of the triangulation τ.
(2) Each proper face of the triangulation is contained in one segment of the set:

$$\{(x+y=i)_{i=1,\ldots,m}, (x=i, y \leq m-i)_{i=1,\ldots,m}, (y=i, x \leq m-i)_{i=1,\ldots,m}\}$$

Choose the following distribution of signs called " Harnack distribution" at each integer points (i,j) of T_m :

(1) If i,j are both even, the integer point (i,j) gets the sign $\epsilon_{(i,j)} = -$
(2) If i or j is odd, the integer point (i,j) gets the sign $\epsilon_{(i,j)} = +$.

The following propositions are deduced from a more general statement due to I.Itenberg [**12**].

Proposition 2.2. [**12**] *Let $m = 2k$ be a positive even integer. The patchworking process applied to T_m with regular triangulation τ and the Harnack distribution of signs at the vertices produces a T-curve which is the Harnack curve $\mathcal{H}_{2k}$ of degree $2k$.*

Proposition 2.3. [**12**]
Let $m = 2k+1$ be a positive even integer. The patchworking process applied to T_m with regular triangulation τ and the Harnack distribution of signs at the vertices produces a T-curve which is the Harnack curve $\mathcal{H}_{2k+1}$ of degree $2k+1$.

Now we shall work out the polynomial entering in this patchworking construction (following essentially Viro's initial method with some modifications necessary for the sequel). We want in particular to stress the recursive character of the patchworking construction of Harnack' s curves.
Fix $m > 0$ integer

(1) An essential element in the construction is the so-called convexifying function. We choose the convexifying function $\nu_m : T_m \to \mathbf{R}$ in such a way that it would be the restriction of ν_{m+1}. Namely, we extend the convexifying function $\nu_m : T_m \to \mathbf{R}$ to a convexifying function ν_{m+1} of the triangulation of $T_{m+1} = T_m \cup D_{m+1,m}$ with

$$D_{m+1,m} = \{(x,y) \in \mathbf{R}^2 \mid 0 \leq x \leq m, 0 \leq y \leq m, x+y \leq m+1\}$$

We shall assume the same notation for ν_m and its extension ν_{m+1}, and for any $m > 1$ denote by ν a convexifying function of the triangulation of T_m.
(2) Now, we construct a polynomial in two variables

$$x_{m,t}(x,y) = \sum_{(i,j) vertices of T_m} \epsilon_{i,j} a_{i,j} x^i y^j t^{\nu(i,j)}$$

where: $\epsilon_{(i,j)}$ is the sign at the vertex (i,j) given by the Harnack distribution of sign, $\nu(i,j)$ is the value of the convexifying function $\nu : T_m \to \mathbf{R}$ on (i,j).

The numbers t and $a_{i,j}$ are parameters which shall be chosen. We shall denote the polynomial $x_{m,t}(x,y)$ by $x_{m(x,y;\vec{a}_m,t)}$ or $x_{m;\vec{a}_m,t}$. Let $\tilde{X}_{m,t}(x_0,x_1,x_2)$ be the homogenization of $x_{m,t}$ obtained by remaining x as x_1, y as x_2, and adding the third variables x_0 so as to obtain :

$$\tilde{X}_{m,t}(x_0,x_1,x_2) = \sum_{(i,j)\ vertices\ of\ T_m} \epsilon_{i,j} a_{i,j} x_1^i x_2^j x_0^{m-i-j} t^{\nu(i,j)}$$

We shall denote the polynomial $\tilde{X}_{m,t}(x_0,x_1,x_2)$ by $\tilde{X}_{m(x_0,x_1,x_2;\vec{a}_m,t)}$ or $\tilde{X}_{m;\vec{a}_m,t}$.

Note that

$$\tilde{X}_{m,t} = x_0.\tilde{X}_{m-1,t} + C_{m,t}$$

where the polynomial $C_{m,t}$ is an homogeneous polynomial of degree in only x_1 and x_2.

Using the patchworking method, one can give an inductive construction, we shall call *T-inductive construction* of some Harnack polynomials.

Assume $\vec{a}_m, t_m$ given in such a way that for any $t \in]0,t_m[$, $\tilde{X}_{m;\vec{a}_m,t}$ is a Harnack polynomial of degree m. Then, the Harnack polynomial of degree $m+1$ is constructed as follows:

(1) **Property 1** *Choose $\vec{c}_{m+1}$ in such a way that for any $t \in]0,t_m[$, singularities of $x_0.\tilde{X}_{m;\vec{a}_m,t}$ do not belong to the curve $\mathcal{C}_{m+1;\vec{c}_{m+1},t}$.*
(2) **Property 2** *Set $\vec{a}_{m+1} = (\vec{a}_m, \vec{c}_{m+1})$. Then, t_{m+1} is chosen as follows:*
   (a) $t_{m+1} \leq t_m$
   (b) *t_{m+1} is the biggest $\tau \leq t_m$ such that for any $t \in]0,\tau[$ the polynomial $\tilde{X}_{m+1;\vec{a}_{m+1},\tau} = x_0.\tilde{X}_{m;\vec{a}_m,\tau} + C_{m+1;\vec{c}_{m+1},\tau}$ is a Harnack polynomial of degree $m+1$.*

Following the preliminary section, consider $\rho^m : \mathbf{R}_+T_m \times U^2_{\mathbf{C}} \to \mathbf{C}T_m$ the natural surjection. Denote $l_m = \{x \geq 0, y \geq 0, x+y = m\}$ the hypotenuse of T_m. The variety $\mathbf{C}T_m$ is homeomorphic to $\mathbf{C}P^2$ with line at infinity L of $\mathbf{C}P^2$ such that $\mathbf{C}l_m \approx L$.

(The real line $\mathbf{R}L \subset \mathbf{R}P^2$ is obtained from the square $\{(x,y) \mid |x|+|y| = m\}$ by gluing of the sides by s the central symmetry with center $0 = (0,0)$).

From the Patchworking construction of Harnack curves it follows that:

Corollary 2.4. *The Harnack curve $\mathcal{H}_m$ of degree m obtained by patchworking process applied to T_m with triangulation τ and Harnack distribution of signs at the vertices intersects the line at infinity $L \approx \mathbf{C}l_m$ of $\mathbf{C}P^2 \approx \mathbf{C}T_m$ in m real points. Thus, the resulting curve $\mathcal{H}_m$ is a curve of type $\mathcal{H}$.*

Let us denote $a_1,...,a_m$ the real points of the intersection

$$\mathbf{R}\mathcal{H}_m \cap \mathbf{R}L = \{a_1,...,a_m\}$$

proof:

First of all, recall that in the patchworking construction of $\mathcal{H}_m$, the real line at infinity $\mathbf{R}L$ of $\mathbf{R}P^2$ is obtained from the square $\{(x,y) \in \mathbf{R}^2 || x| + |y| = m\}$ by gluing of the sides by s the central symmetry with center $0 = (0,0)$. Furthermore, according to the patchworking construction, the real point set of the Harnack curve separates any two consecutive integer points of the square $\{(x,y) \in \mathbf{R}^2 || x| + |y| = m\}$ with different signs. Thus, the corollary follows immediately from the Harnack distribution of signs on T_m and its extension to its symmetric copies.Q.E.D

Moreover, one can make the following remark:

REMARK 2.5. Harnack curves are curves of type I, in other words their real set of points are orientable. Let $\mathcal{H}_m$ be the Harnack curve of degree m. Assume $\mathbf{R}\mathcal{H}_m$ given with an orientation. There is only one orientation of $\mathbf{R}L$ compatible with the deformation of $\mathcal{H}_m \cup L$ to $\mathcal{H}_{m+1}$ and an orientation of $\mathbf{R}\mathcal{H}_{m+1}$. In such a way, we obtain the relative orientation of the connected components of the real point set of the Harnack curves.

CHAPTER 2

Morse-Petrovskii's Theory of Harnack curves

In this chapter, at first we shall study critical points of Harnack polynomials from a view-point initiated by Petrovskii. This investigation is analogous to those of Morse on critical points of functions.
Then, using rigid isotopy classification of Harnack curves, we shall construct deformation of Harnack polynomial and deduce a description up to conj-equivariant isotopy of $\mathbf{C}P^2$ of the complex set of points of Harnack curve (see proposition 3.7 of Chapter 2 and theorem 3.9 of Chapter 2).
We shall divide this chapter into three sections.
In the first section, we start with proposition 1.1 of Chapter 2 in which we prove that up to slightly modify coefficients of a Harnack polynomial, the number of its critical points depends only on its degree. Then, in proposition 1.4 of Chapter 2, we precise, up to change the system of projective coordinates if necessary, the number of critical points of index 1 with positive and respectively negative critical value of such modified Harnack polynomials. Such results may be obtained from the Viro's patchworking method for T-Harnack polynomials, it is explicitely explained in Shustin's paper [**17**].
In the second section, we consider only real points set of Harnack curves. In Theorem 2.1, we prove that isotopy also implies rigid isotopy for Harnack curves $\mathcal{H}_m$ of degree m. In the third section, we define in Proposition 3.1 of Chapter 2, a deformation of $\mathcal{H}_m$, to a singular irreducible curve of which singular points are critical points of index 1 with positive critical value of $\mathcal{H}_m$.
The main result of this chapter is given in the third and last section. It provides link between properties of L-curves ([**8**]) and the Harnack curves. In Theorem 3.9 of Chapter 2, we present any Harnack curve $\mathcal{H}_m$ of degree m up to conj-equivariant isotopy of $\mathbf{C}P^2$ as follows. Outside a finite number of 4-balls $B(a_i)$ globally invariant by complex conjugation, $\mathcal{H}_m$ splits in m non-intersecting projective lines minus their intersection with the $B(a_i)$; inside any 4-ball $B(a_i)$ it is the perturbation of a crossing.

1. Critical points of Harnack polynomials

Recall a Harnack polynomial of degree m is a non-singular homogeneous polynomial in three variables such that its set of zeros has real scheme:

for even $m = 2k$:

$$\left\langle 1\left\langle \frac{(k-1)(k-2)}{2}\right\rangle \sqcup \frac{3k(k-1)}{2}\right\rangle$$

for odd $m = 2k + 1$:

$$\left\langle J \sqcup k(2k-1)\right\rangle$$

In this section, we study the critical points and the level surfaces of Harnack polynomials.

Given $x_0 := 0$ the line at infinity and $R(x_0, x_1, x_2)$ an homogeneous polynomial of degree m, we call affine restriction $r(x, y)$ of $R(x_0, x_1, x_2)$ the unique polynomial such that $R(x_0, x_1, x_2) = x_0^m.r(x_1/x_0, x_2/x_0)$. In such a way, we consider $\mathbf{C}P^2$ as the completion of $\mathbf{C}^2$ with $x_0 := 0$ the line at infinity. We call critical point of $r(x, y)$,(and by misuse critical point of $R(x_0, x_1, x_2)$), any point (x_0, y_0) (finite or not) such that $r_x(x_0, y_0) = 0$, $r_y(x_0, y_0) = 0$.

We shall consider generic polynomials. To be more precise, let us call an homogeneous non-singular real polynomial $R(x_0, x_1, x_2)$ with affine restriction $r(x, y)$ *regular* if :

(1) none of critical points of $r(x, y)$ lies on the line at infinity
(2) any two different real critical points $r(x, y)$ have different critical values.

From ([**16**], p.359 Lemma 2), given a polynomial of a smooth algebraic curve, one can always perturb its coefficients so as to have 1) and 2) as above.

Bringing together Lemma 0.11 of Chapter 0 and Petrovskii's Lemma ([**16**], p.359 Lemma 2), it follows that the set of regular polynomials is open and dense in the set of all polynomials.

We shall call *polynomial of type* $\mathcal{H}$ a regular Harnack polynomial of degree m giving a Harnack curve $\mathcal{H}_m$ which intersects the line at infinity of $\mathbf{R}P^2$ in m real distincts points. We shall call *polynomial of type* $\mathcal{H}^0$ a polynomial of type $\mathcal{H}$ such that the line at infinity of $\mathbf{R}P^2$ intersects the Harnack curve $\mathcal{H}_m$ of degree m in m real distinct points which belong to the same connected component of $\mathbf{R}\mathcal{H}_m$: in case $m = 2k + 1$ the real connected component of $\mathcal{H}_{2k+1}$ homeomorphic to the projective line; in case $m = 2k$ the non-empty oval of $\mathcal{H}_{2k}$.

Let us start by the proposition 1.1 of Chapter 2 in which we find the number of critical points of all indices of any regular Harnack polynomial of degree m.

For a regular polynomial of homogeneous degree m, $R(x_0, x_1, x_2)$ with affine restriction $r(x, y)$,$R(x_0, x_1, x_2) = x_0^m.r(x_1/x_0, x_2/x_0)$, let us denote by $c_i(R)$ the number critical points of index i of $r(x, y)$ and set $c(R) = (c_0(R), c_1(R), c_2(R))$.

PROPOSITION 1.1. *Let $B_m(x_0, x_1, x_2)$ be a Harnack polynomial of degree m of type $\mathcal{H}^0$.*
Then, up to change the sign of B_m:

(1) $c_0(B_m) + c_1(B_m) + c_2(B_m) = (m - 1)^2$
(2) (a) *For even $m = 2k$,*

$$c(B_{2k}) = \Big(\frac{(k-1)(k-2)}{2}, k(2k-1), \frac{3k(k-1)}{2}\Big)$$

(b) *For odd $m = 2k + 1$,*

$$c(B_{2k+1}) = \Big(\frac{k(k-1)}{2}, k(2k+1), \frac{k(3k-1)}{2}\Big)$$

proof:

Let $b_m(x, y)$ be the affine restriction of the polynomial $B_m(x_0, x_1, x_2)$. For any oval which does not intersect the line at infinity, consider the disc with boundary

the oval. In case of even oval, the gradient of the affine polynomial $b_m(x, y)$ points inward. Therefore, the maximum of the polynomial b_m on the disc is in the interior of the oval and is necessary a critical point of index 2 of b_m.

In case of odd oval, the gradient of the affine polynomial $b_m(x, y)$ points outward. Therefore, the minimum of the polynomial b_m on the disc is in the interior of the oval and is necessarily a critical point of index 0 of b_m. This implies the following inequalities (up to change the sign of B_m and in case of even degree curve according to the convention. -ovals not lying within other ovals or lying inside an even number number of consecutive ovals are outer, while ovals lying within an odd number of ovals are inner-)

for even $m = 2k$:

$$c_0 \geq \frac{(k-1)(k-2)}{2}, \quad c_2 \geq \frac{3k(k-1)}{2}$$

for odd $m = 2k + 1$:

$$c_0 \geq \frac{k(k-1)}{2}, \quad c_2 \geq \frac{k(3k-1)}{2}$$

Furthermore, since $\mathcal{H}_m$ intersects the line at infinity in m real points we have $c_0 - c_1 + c_2 = 1 - m$ (see [**7**]). Thus, it follows from the inequalities above that for even $m = 2k$:

$$c_1 \geq k(2k-1) \tag{3}$$

for odd $m = 2k + 1$:

$$c_1 \geq k(2k+1) \tag{4}$$

and thus $c_0 + c_1 + c_2 \geq (m-1)^2$.

From the Bezout's theorem, the number of non-degenerate critical points of a homogeneous polynomial of degree m does not exceed $(m-1)^2$. Therefore $c_0 + c_1 + c_2 = (m-1)^2$ and the inequalities 3 and 4 are equalities.

Q.E.D.

Our next aim is to find the number of critical points of indice 1 with negative (resp, positive) critical value of a any polynomial of type $\mathcal{H}^0$. This will be done in Proposition 1.4 of Chapter 2.

For a regular polynomial, let us denote by $c_1^-(R)$ (resp, $c_1^+(R)$) the number of critical points of index 1 of R of negative (resp, positive) critical value. Besides, denote by $c_1'(R)$ the number of critical points of index 1 of R with positive critical value c_0 such that as c increases from $c_0 - \epsilon$ to $c_0 + \epsilon$ the number of connected components of $M_c = \{(x, y) \in \mathbf{R}^2 \mid r(x, y) > c\}$ of which boundary contains the line at infinity increases by 1. We shall denote S'_{2k} the set constituted by the $c_1'(B_{2k})$ critical points of index 1 of a polynomial B_{2k} of type $\mathcal{H}^0$.

Our proof uses Petrovskii's theory introduced in [**16**]. It is based on the consideration of the lines $b_m(x, y) = c$ when c crosses the critical value of a polynomial B_m of type $\mathcal{H}^0$.

These last investigations are analogous to Morse theory.

Let us recall Petrovskii's Lemmas implicit in [**16**]
(p.360-361).

First, introduce definitions.

Define $\mathbf{R}P^2$ as a Möbius strip and a disc D^2 glued along their boundaries. The core of the Möbius strip intersects the line at infinity in a finite number of points. Call "odd component of the affine plane $\mathbf{R}^2$" the core of the Möbius strip minus its intersection with the line at infinity.

LEMMA 1.2. [**16**] *Let A a curve of degree m given by a regular polynomial $R(x_0, x_1, x_2) = x_0^m r(x_1/x_0, x_2/x_0)$ with $(m-1)^2$ critical points. Let (x_0, y_0) be a finite real critical point of $r(x, y)$. Setting $r(x_0, y_0) = c_0$, assume that $r^{-1}[c_0 - \epsilon, c_0 + \epsilon]$ contains no critical point other than (x_0, y_0).*

Assume the Hessian $H(x_0, y_0) < 0$, (i.e (x_0, y_0) is a critical point of index 1).

(*) Assume $r(x, y)$ is of even degree, and c decreases from $c_0+\epsilon$ to $c_0-\epsilon$. Then there are three possibilities:

(1) *When c takes value c_0, one outer oval touches another oval (outer or inner), then one outer oval disappears.*
(2) *When c takes value c_0, one oval (outer or inner) touches itself.*
    (a) *If one of the components $r(x, y) > r(x_0, y_0)+\epsilon$; $r(x, y) < r(x_0, y_0)-\epsilon$ contains the odd component of $\mathbf{R}^2$, then one inner oval appears.*
    (b) *Otherwise, one outer oval becomes inner. (This case may present itself at most once as c varies from $+\infty$ to $-\infty$.)*

(*) Assume r(x,y) is of odd degree and c decreases from $c_0 + \epsilon$ to $c_0 - \epsilon$. Then there are three possibilities:

(1) *When c takes value c_0, one outer oval touches an other oval or the one-side component, then the outer oval disappears.*
(2) *When c takes value c_0, one oval (outer or inner) touches itself then one inner oval appears.*
(3) *When c takes value c_0, one zero oval or the odd-component of the curve touches itself or an other zero oval, then there are three possibilities:*
    Consider the regions G_i $i = 1, ..., r$ where $r(x, y) > c_0$ which contain segment of the line at infinity of $\mathbf{R}P^2$ on their boundary.
    (a) *the boundary of one region G_i touches itself (in one point which does not belong to the line at infinity), then one inner oval appears.*
    (b) *one region G_i touches the boundary (not contained in the line at infinity) of another such regions G_j then the regions coalesce and a zero-oval disappears or a zero oval appears.*

The next Lemma is implicit in([**16**], Lemma 4, p.361)

LEMMA 1.3. *(line at infinity ([**16**]) Let A a curve of degree m given by a regular polynomial $R(x_0, x_1, x_2) = x_0^m.r(x_1/x_0, x_2/x_0)$. with $(m-1)^2$ critical points.*

Assume furthermore that A intersects the real line at infinity in k ($k > 1$) distinct points. Let $M_c = \{(x, y) \in \mathbf{R}^2 \mid r(x, y) > c\}$ Let c_M be the maximal critical value, and c_m be the minimal critical value.

(1) *For $c > c_M$, M_c has k connected components.*
(2) *For $c < c_m$, M_c is simply connected. Two of the connected components above can coalesce only when $r(x, y)$ passes a critical value c of index 1.*

PROPOSITION 1.4. *Let $m \geq 0$ and $B_m(x_0, x_1, x_2)$ be a Harnack polynomial of type $\mathcal{H}^0$. Then, up to change the sign of B_m:*

(1) *for even $m = 2k$;*

$$c_1^-(B_{2k}) = \frac{k(3k-1)}{2}$$

$$c_1^+(B_{2k}) = \frac{k(k-1)}{2}, \quad c_1'(B_{2k}) = k-1$$

(2) *for odd $m = 2k+1$*

$$c_1^-(B_{2k+1}) = \frac{k(3k+3)}{2}$$

$$c_1^+(B_{2k+1}) = \frac{k(k-1)}{2}, \quad c_1'(B_{2k+1}) = 0$$

proof :

The proof of Proposition 1.4 of Chapter 2 is based on the Lemma 1.2 of Chapter 2 and Lemma 1.3 of Chapter 2.

Let $\mathcal{H}_m$ be the Harnack curve of degree m given by the polynomial $B_m(x_0, x_1, x_2) = x_0^m b_m(x_1/x_0, x_2/x_0)$ of type $\mathcal{H}^0$. Consider the pencil of curves given by polynomials $x_0^m(b_m(x_1/x_0, x_2/x_0) - c)$ with $c \in \mathbf{R}$.

(1) Let c decrease from 0 to $c_m - \epsilon$, $\epsilon > 0$, the inner ovals shrink and disappear, the outer disappear in a unique oval. It follows from the Lemma 1.3 of Chapter 2 that the set $M_{c_m-\epsilon}$ is simply connected. Furthermore, from Lemma 1.2 of Chapter 2 : one outer oval can touch another outer oval and then disappears as c decreases from $c_0 + \epsilon$ to $c_0 - \epsilon$, if and only if c_0 is a critical value of index 1 of the polynomial b_m.

(*)Let $\mathcal{H}_{2k}$ be the Harnack curve of even degree $2k$

When c decreases from 0 to $c_m - \epsilon$, outer ovals expand, then coalesce and finally disappear in a unique oval. Moreover, from Lemma 1.2 of Chapter 2, to each coalescence of a outer oval $\mathcal{O}$ with an other outer oval is associated a critical point (x_0, y_0) of index 1 with $b_{2k}(x_0, y_0) = c_0$; $c_0 \in]c_m, 0[$ such that as c decreases from $c_0 + \epsilon$ to $c_0 - \epsilon$ the oval $\mathcal{O}$ disappears. Denote S_{2k}^- the set of these critical points. The curve $\mathcal{H}_{2k}$ has $\frac{3k(k-1)}{2}$ outer ovals and $(k+1)$ connected components of $M_0 = \{(x,y) \in \mathbf{R}^2 | b_{2k}(x,y) > 0\}$ contain a segment of the line at infinity of $\mathbf{R}P^2$ on their boundary. Therefore, the number of critical points of S_{2k}^- is

$$c_1^-(B_{2k}) \geq \frac{3k(k-1)}{2} - 1 + (k+1) = \frac{k(3k-1)}{2} \tag{5}$$

(*)Let $\mathcal{H}_{2k+1}$ be the Harnack curve of odd degree $2k+1$

When c decreases from 0 to $c_m - \epsilon$, outer ovals and regions G_i where $b_{2k+1} > c_0$ which intersect the line at infinity of $\mathbf{R}P^2$ on their boundary, expand, then coalesce, and finally disappear in a simply connected region.

Moreover, from Lemma 1.2 of Chapter 2, one can associate to each coalescence of one outer oval $\mathcal{O}$ with an other outer oval or with one of the regions G_i a critical point (x_0, y_0) of index 1 with $b_{2k}(x_0, y_0) = c_0$;

$c_0 \in]c_m, 0[$ with the property that $\mathcal{O}$ disappears as c decreases from $c_0+\epsilon$ to $c_0-\epsilon$.

Denote S^-_{2k+1} the set of these critical points. The curve $\mathcal{H}_{2k+1}$ has $\frac{k(3k-1)}{2}$ outer ovals and $(2k+1)$ connected components $M_0 = \{(x,y)\in \mathbf{R}^2 | b_{2k+1}(x,y) > 0\}$ which contain a segment of the line at infinity of $\mathbf{R}P^2$ on their boundary. Therefore, the number of critical points of S^-_{2k+1} is:

$$c_1^-(B_{2k+1}) \geq \frac{k(3k-1)}{2} + 2k = \frac{k(3k+3)}{2} \tag{6}$$

(2) Let c increase from 0 to $c_M+\epsilon$, then the outer oval shrink and disappear. Moreover, from the Lemma 1.3 of Chapter 2, the set $M_{c_M+\epsilon}$ has m components.

(*)Let $\mathcal{H}_{2k}$ be the Harnack curve of even degree $2k$

The set M_0 has $(2k-(k-1))$ components intersecting the line at infinity; the curve $\mathcal{H}_{2k}$ has $\frac{(k-1)(k-2)}{2}$ inner ovals. It is easy to deduce from the Lemma 1.2 of Chapter 2 that when c increases from 0 to $c_M+\epsilon$, the inner ovals expand, touch the non-empty outer oval or another inner oval and then disappear. In other words, when c decreases from $c_M+\epsilon$ to 0, $\frac{(k-1)(k-2)}{2}$ inner ovals appear. Hence, from Lemma 1.2 of Chapter 2(2a), one can associate to each inner oval $\mathcal{O}$ a critical point (x_0, y_0) of index 1 with $b_{2k}(x_0,y_0) = c_0$; $c_0 \in]0, c_M[$ with the property that $\mathcal{O}$ disappears as c increases from $c_0-\epsilon$ to $c_0+\epsilon$.

Furthermore, the set $M_{c_M+\epsilon}$ has $2k$ components. Thus, from the Lemma 1.3 of Chapter 2, c has passed at least $(k-1)$ critical values of index 1 in his way from 0 to $c_M+\epsilon$. We denote by S'_{2k} the set of critical points associated to these critical values.

Denote S^+_{2k} the set of critical points of index 1 (x_0,y_0) such that $b_{2k}(x_0,y_0) = c_0$; $c_0 \in]0, c_M[$. Therefore, the number of critical points of S^+_{2k} is:

$$c_1^+(B_{2k}) \geq \frac{(k-1)(k-2)}{2} + (k-1) = \frac{k(k-1)}{2} \tag{7}$$

(*)Let $\mathcal{H}_{2k+1}$ be the Harnack curve of odd degree $2k+1$.

The set M_0 has $2k+1$ components intersecting the line at infinity; the curve $\mathcal{H}_{2k+1}$ has $\frac{k(k-1)}{2}$ inner ovals. When c increases from 0 to $c_M+\epsilon$; inner ovals expand, then touch the one-side component or another inner oval, and finally disappear.

Moreover, the set $M_{c_M+\epsilon}$ has $2k+1$ components.

From Lemma 1.2 of Chapter 2 (3), one can associate to each inner oval $\mathcal{O}$ of $\mathcal{H}_{2k+1}$ a critical point (x_0,y_0) of index 1 with $b_{2k+1}(x_0,y_0) = c_0$; $c_0 \in]0, c_M[$ with the property that $\mathcal{O}$ disappears as c increases from $c_0-\epsilon$ to $c_0+\epsilon$.

Denote S^-_{2k+1} the set of these critical points. The curve $\mathcal{H}_{2k+1}$ has $\frac{k(k-1)}{2}$ inner ovals. Hence, the number of critical points of S^+_{2k+1} is:

$$c_1^+(B_{2k+1}) \geq \frac{k(k-1)}{2} \tag{8}$$

Hence, according to Proposition 1.1 of Chapter 2, inequalities 5, 6, 7, 8 are equalities. This implies the Proposition 1.4 of Chapter 2. Q.E.D In particular, this method provides curves with real scheme:
for even $m = 2k$

2. Harnack Curves from a real viewpoint-Rigid Isotopy Classification-

Passing from polynomials to real set of points, it follows that real algebraic curves form a real projective space of dimension $\frac{m(m+3)}{2}$. We shall denote this space by the symbol $\mathbf{R}\mathcal{C}_m$ and by $\mathbf{R}\mathcal{D}_m$ the subset of $\mathbf{R}\mathcal{C}_m$ corresponding to real singular curves. We call a path in the complement $\mathbf{R}\mathcal{C}_m \backslash \mathbf{R}\mathcal{D}_m$ of the discriminant hypersurface in $\mathbf{R}\mathcal{C}_m$ a *rigid isotopy* of real point set of nonsingular curves of degree m. The classification of real point set of curves of degree ≤ 4 up to rigid isotopy is known since the 19^{th} century. It was completed for curves of degree ≤ 5 and ≤ 6 at the end of the seventies. Up to rigid isotopy a curve of degree ≤ 4 is determined by its real scheme; up to rigid isotopy a curve of degree 5 or 6, is determined by its real scheme and its type.

Let us recall that we denote by $\mathcal{H}_m$ and call Harnack curve any curve with real scheme:
- for even $m = 2k$

$$\left\langle 1\left\langle \frac{(k-1)(k-2)}{2}\right\rangle \sqcup \frac{3k(k-1)}{2}\right\rangle$$

- for odd $m = 2k+1$

$$\left\langle J \sqcup k(2k-1)\right\rangle$$

The main result of this section is given in the Theorem 2.1 of Chapter 2 where we establish the rigid isotopy classification of real point set of Harnack curves $\mathbf{R}\mathcal{H}_m$. Harnack curves of degree $i \leq 6$, as any M-curve of degree ≤ 6, are rigidly isotopic. In Theorem 2.1 of Chapter 2, we extend this property to Harnack curves of arbitrary degree. Precisely, we prove that isotopy also implies rigid isotopy for real point set of Harnack curves. This section until its end is devoted to the proof of this result.

Rigid Isotopy Classification Theorem

THEOREM 2.1. *Harnack curve $\mathcal{H}_m$ of degree m are rigidly isotopic.*

The proof of Theorem 2.1 of Chapter 2 is based on a modification of Harnack polynomials. Let us call *regular modification* of a polynomial any modification on its coefficients with the property that real point set of curves of the modified polynomial and the initial polynomial are rigidly isotopic.
(In particular, given a polynomial of a smooth algebraic curve, the modification of its coefficients such that the modified polynomial is a regular polynomial (i.e none of its critical points lies on the line at infinity, any two different critical points have distinct critical values) is obviously regular.)
Let $\mathcal{A}$ and $\mathcal{B}$, be two smooth curves such that the union $\mathcal{A} \cup \mathcal{B}$ is a singular curve all

of whose singular points are crossings. Denote by $\mathcal{H}$ the set of curves which result from the classical deformation of $\mathcal{A} \cup \mathcal{B}$. We shall say that a curve is deduced from *deformation* of $\mathcal{A} \cup \mathcal{B}$ if it is rigidly isotopic to a curve $\mathcal{C}$ of the set $\mathcal{H}$.
Any two Harnack curves $\mathcal{H}_m$ constructed from the Harnack's method are rigidly isotopic.(We recall Harnack's method and propose a proof of this statement in the Appendix) The proof of the Theorem 2.1 of Chapter 2 is based on a regular modification of Harnack polynomials. From properties of this regular modification, we shall deduce that any Harnack curve $\mathcal{H}_m$ is, up to rigid isotopy, constructed from the Harnack's method. In this way, we get the rigid isotopy Theorem 2.1 of Chapter 2.

Harnack curve of type $\mathcal{H}^0$. Let us recall that we call *polynomial of type* $\mathcal{H}$ a regular Harnack polynomial of degree m giving a Harnack curve $\mathcal{H}_m$ which intersects the line at infinity $\mathbf{R}L$ of $\mathbf{R}P^2$ in m real distinct points. We shall call *polynomial of type* $\mathcal{H}^0$ (relatively to L) a polynomial of type $\mathcal{H}$ such that the line at infinity L of $\mathbf{R}P^2$ intersects the Harnack curve $\mathcal{H}_m$ of degree m in m real distinct points which belong to the same connected component of $\mathbf{R}\mathcal{H}_m$: in case $m = 2k$, the non-empty oval of $\mathcal{H}_{2k}$; in case $m = 2k+1$, the odd component of $\mathcal{H}_{2k+1}$. The odd component of $\mathcal{H}_{2k+1}$ is divided into $2k+1$ arcs which delimit a region of which boundary contains a segment of the line at infinity. The line at infinity is chosen such that only one of this region contains ovals of $\mathcal{H}_{2k+1}$. We say that a curve is of type $\mathcal{H}^0$ if its polynomial is of type $\mathcal{H}^0$. In particular, Harnack curves constructed from the Harnack's method (see Appendix) are curves of type $\mathcal{H}^0$.

The main result of this subsection is the Theorem 2.2 of Chapter 2 where we give the rigid isotopy classification of curves of type $\mathcal{H}^0$.

THEOREM 2.2. *Harnack curve $\mathcal{H}_m$ of degree m and type $\mathcal{H}^0$ are rigidly isotopic.*

Let $\mathcal{H}_m$ be a Harnack curve with polynomial $B_m(x_0, x_1, x_2)$ of type $\mathcal{H}^0$. Denote by $C_i(x_1, x_2)$ the unique homogeneous polynomial of degree i in the variables x_1, x_2 such that: $B_m(x_0, x_1, x_2) = x_0^{m-1}.B_1(x_0, x_1, x_2) + \Sigma_{i=2}^{m} x_0^{m-i}.C_i(x_1, x_2)$.
Let $b_m(x, y)$ be the affine polynomial associated to $B_m(x_0, x_1, x_2)$,

$$B_m(x_0, x_1, x_2) = x_0^m.b_m(x_1/x_0, x_2/x_0)$$

Let $b_i(x, y)$ be the truncation of $b_m(x, y)$ on the monomials $x^\alpha.y^\beta$ with $0 \leq \alpha+\beta \leq i$ and $B_i(x_0, x_1, x_2) = x_0^i.b_i(x_1/x_0, x_2/x_0)$ be the homogeneous polynomial associated to b_i. We shall denote by $\mathcal{B}_i$ the curve with polynomial $B_i(x_0, x_1, x_2)$.

Consider the norm in the vector space of polynomials

$$||\sum a_{i,j} x_1^i x_2^j|| = max\{|a_{i,j}| \;|(i,j) \in \mathbf{N}^2\}$$

(Given $A_m(x_0, x_1, x_2)$ an homogeneous polynomial,

$$||A_m(x_0, x_1, x_2)|| = ||A_m(1, x_1, x_2)||$$

)

The Theorem 2.2 of Chapter 2 may be also formulated as follows:

PROPOSITION 2.3. *On the assumption that $B_m(x_0, x_1, x_2)$ is of type $\mathcal{H}^0$ relatively to the line at infinity $x_0 = 0$, up to regular modification of $B_m(x_0, x_1, x_2)$, for any $B_i(x_0, x_1, x_2)$, $i \geq 1$, we have:*

(1) *$B_i(x_0, x_1, x_2)$ is smooth of type $\mathcal{H}^0$*
(2) *none of the critical points of $B_i(x_0, x_1, x_2)$ belongs to the line at infinity; for any critical point $(1, x_{0,1}, x_{0,2}) \in \mathbf{R}P^2$ of $B_i(1, x_1, x_2) = b_i(x_1, x_2)$ its representative in S^2 $\frac{1}{(1+x_{0,1}^2+x_{0,2}^2)^{1/2}}(1, x_{0,1}, x_{0,2})$ is such that:* $B_i(\frac{1}{(1+x_{0,1}^2+x_{0,2}^2)^{1/2}}(1, x_{0,1}, x_{0,2})) \notin [-\Sigma_{j=i+1}^m ||C_j||, \Sigma_{j=i+1}^m ||C_j||]$

On the assumption that B_m is of type $\mathcal{H}^0$, according to proposition 1.1 of Chapter 2, the following equalities are verified:
for even $m = 2k$: $c(B_{2k}) = (\frac{(k-1)(k-2)}{2}, k.(2k-1), \frac{3k.(k-1)}{2})$
and for odd $m = 2k+1$: $c(B_{2k+1}) = (\frac{k.(k-1)}{2}, k.(2k+1), \frac{k.(3k-1)}{2})$

proof:
Our proof is based on Morse Lemma and Petrovskii's theory. We shall proceed by descending induction on the degree i of B_i. Let us assume that assumptions (1) and (2) of Proposition 2.3 of Chapter 2 are satisfied for B_n with $i+1 \leq n \leq m$, and prove that B_i also satisfies (1) and (2).
On these assumptions, we shall prove that, the Harnack curve $\mathcal{H}_{i+1}$ with polynomial B_{i+1} is deduced from deformation of $\mathcal{B}_i \cup L$,$B_{i+1}(x_0, x_1, x_2) = x_0.B_i(x_0, x_1, x_2) + C_{i+1}(x_1, x_2)$, where $L := \{x_0 = 0\}$ and B_i is of type $\mathcal{H}^0$ relatively to L.

I) Let us in a first part study curves $\mathcal{B}_i$ of degree ≥ 4. Up to regular modification of the Harnack polynomial B_{i+1} one can always assume that B_i is smooth.

Introduction

Consider a real projective line $\mathbf{R}L \subset \mathbf{R}P^2$; its tubular neighborhood in $\mathbf{R}P^2$ is homeomorphic to a Möbius band. The core of the Möbius band, i.e the real projective line $\mathbf{R}L$, is the circle with framing ± 1. In such a way, one can identify points of $\mathbf{R}L$ with points of two halves (oriented) circles. Halves intersect each other in two points we shall call *extremities*. We shall denote $\mathbf{R}L^+$ and $\mathbf{R}L^-$ the halves of $\mathbf{R}L$. On each half $\mathbf{R}L^\pm$ of $\mathbf{R}L$ one can consider an oriented tubular fibration. We shall denote by $\mathcal{M}^\pm$ the half of the Möbius band which is the tubular neighborhood of $L^\pm$. In such a way, the boundary of the half of Möbius band $\mathcal{M}^\pm$ contains the union of two real projective lines $\mathbf{R}L_1^\pm$, $\mathbf{R}L_2^\pm$. The half $\mathbf{R}L_i^\pm$ of $\mathbf{R}L_i$, $i \in \{1, 2\}$, is the image of a smooth section of the tubular oriented fibration of a tubular neighborhood of $\mathbf{R}L^\pm$; $\mathcal{M}^- \cup \mathcal{M}^+ = \mathcal{M}$, $\partial\mathcal{M} \supset \mathbf{R}L_1^\pm \cup \mathbf{R}L_2^\pm$.

Let $\mathcal{H}_m$ be the Harnack curve of degree m with polynomial B_m of type $\mathcal{H}^0$ relatively to L. We shall study $\mathbf{R}\mathcal{H}_m$ in a tubular neighborhood $\mathcal{M}$ of the line at infinity $\mathbf{R}L$.
Let $\mathcal{A}$ be the set of arcs of $\mathbf{R}\mathcal{H}_m$ bounding regions which contain a segment of the line at infinity on their boundary and do not contain ovals of $\mathbf{R}\mathcal{H}_m$. We shall say that a critical point of index 1 is *associated* to an arc $\gamma \in \mathcal{A}$ of $\mathbf{R}\mathcal{H}_m$ if there exists p a critical point of b_m with critical value c_0 such that as c varies from 0 to c_0, the region which contains γ on its boundary varies in such a way that for $c = c_0$ it touches an other arc γ'. The arc γ' is said *associated* to the arc γ and the pair (γ, γ') is said associated to the critical point p. We shall denote $\mathcal{A}'$ the set of arcs γ'.

From the study of the Petrovskii's pencil $x_0^m.(b_m(x_1/x_0, x_2/x_0)-c)$, $c \in \mathbf{R}$ over $\mathcal{H}_m$ with $L := x_0 = 0$, we shall define a set $\mathcal{P}$ of critical points of b_m and a set of arcs of $\mathbf{R}\mathcal{H}_m$ associated to $\mathcal{A}$.
The set $\mathcal{A}$ is the image of a smooth section s of a tubular fibration of $\mathbf{R}L^+$ minus 2 points. The proof of Theorem 2.2 of Chapter 2 is based on the characterization (up to regular deformation of B_m) of a subset of $\mathbf{R}\mathcal{H}_m$ which is the image of the extension of s to a smooth section of a tubular fibration of $\mathbf{R}L$ minus a finite number of points.
We shall proceed as follows. Using Morse Lemma and Petrovskii's theory, we shall define a set of arcs $\mathcal{D}$ of $\mathbf{R}\mathcal{H}_m$ with the property that up to regular modification of B_m the line at infinity $\mathbf{R}L$ divides any arc $\xi \in \mathcal{D}$ into two halves which belong respectively to $\mathbf{R}\mathcal{B}_{m-1}$ and $\mathbf{R}\mathcal{B}_m \backslash \mathbf{R}\mathcal{B}_{m-1}$ where $\mathcal{B}_{m-1}$ is the Harnack curve of degree $m-1$. In such a way, we shall deduce that the Harnack curve $\mathcal{H}_m$ is deduced from deformation of $\mathcal{H}_{m-1} \cup L$ $B_m(x_0,x_1,x_2) = x_0.B_{m-1}(x_0,x_1,x_2) + C_m(x_1,x_2)$, where $L := \{x_0 = 0\}$ and B_{m-1} is of type $\mathcal{H}^0$.

Let us distinguish in parts **I.i)** and **I.ii)** curves of even and odd degree.

I.i) -Harnack curves of even degree-

Let $\mathcal{H}_{2k}$ be a Harnack curve of degree $2k$ with polynomial $B_{2k}(x_0,x_1,x_2)$ of type $\mathcal{H}^0$ relatively to the line at infinity $L := x_0 = 0$. We shall prove the following Proposition 2.4 of Chapter 2.

PROPOSITION 2.4. *Let $\mathcal{H}_{2k}$ be a Harnack curve of degree $2k$ with polynomial $B_{2k}(x_0,x_1,x_2)$ of type $\mathcal{H}^0$ relatively to the line at infinity $L := x_0 = 0$. Up to regular modification of B_{2k}, $B_{2k-1}(x_0,x_1,x_2)$ is also of type $\mathcal{H}^0$ relatively to L.*

proof:

The part I.i) until its end is devoted to the proof of Proposition 2.4 of Chapter 2. The non-empty oval of $\mathcal{H}_{2k}$ is divided into $2k$ arcs which delimit a region which contains a segment of the line at infinity on its boundary. Denote $\mathcal{A}$ the set of the $2k-1$ arcs which delimit a region which does not contain oval of $\mathbf{R}\mathcal{H}_{2k}$; $k-1$ (resp, k) of them delimit regions $\{x \in \mathbf{R}^2 | b_{2k}(x) < 0\}$ (resp, $\{x \in \mathbf{R}^2 | b_{2k}(x) > 0\}$). The union of arcs of $\mathcal{A}$ is a connected orientable part of the non-empty oval which belongs to a tubular neighborhood of L. The connected surface (with boundary the non-empty oval) obtained from removing the interior of the inner ovals to the interior of the non-empty oval is also orientable. Thus, the orientation of the connected union of the $2k-1$ arcs of $\mathcal{A}$ is induced by a half of $\mathbf{R}L$. We shall denote $\mathbf{R}L^+$ this half. The tubular neighborhood of the line $\mathbf{R}L$ is a Möbius band $\mathcal{M} = \mathcal{M}^- \cup \mathcal{M}^+$ $\partial\mathcal{M}^\pm \supset \mathbf{R}L_1^\pm \cup \mathbf{R}L_2^\pm$. The set $\mathcal{A}$ is the image of a smooth section of a tubular fibration of $\mathbf{R}L^+$ (-precisely, $\mathbf{R}L^+$ minus its extremities, these 2 points belong to the arc of $\mathbf{R}\mathcal{H}_{2k}$ bounding the non-empty region with a segment of the line at infinity on its boundary-)

It is easy to see that there exists an isotopy of $\mathbf{R}P^2$ which pushes the k (oriented) arcs of $\mathcal{A} \cap \mathbf{R}^2$ bounding positive regions $\{x \in \mathbf{R}^2 | b_{2k}(x) > 0\}$ to one positive line $\mathbf{R}L_i^+$, $i \in \{1,2\}$. Let it be $\mathbf{R}L_1^+$. In this way, there exists an isotopy of $\mathbf{R}P^2$ which pushes the $k-1$ (oriented) arcs of $\mathcal{A} \cap \mathbf{R}^2$ bounding negative regions $\{x \in \mathbf{R}^2 | b_{2k}(x) < 0\}$ to $\mathbf{R}L_2^+$.

Let us prove that one can associate to the set of $2k-1$ arcs $\mathcal{A}$ a set of critical points $\mathcal{P}$ of index 1 of b_{2k} and a set of arcs $\mathcal{A}'$ of $\mathbf{R}\mathcal{H}_{2k}$.

To this end, we shall study the pencil of curves $x_0^{2k}.(b_{2k}(x_1/x_0, x_2/x_0) - c)$, $c \in \mathbf{R}$, over $\mathcal{H}_{2k}$.
Let us recall that (see Petrovskii's Lemmas 1.2 of Chapter 2 1.3 of Chapter 2) as c decreases from 0 positive regions and positive ovals expand. As c decreases from 0 the number of regions G_i of $\{x \in \mathbf{R}^2 | b_{2k}(x) > 0\}$ which contain a segment of the line at infinity of $\mathbf{R}P^2$ on their boundary decreases from $k+1$ to 1.
As c increases from 0 the number of regions G_i of $\{x \in \mathbf{R}^2 | b_{2k}(x) > 0\}$ which contain a segment of the line at infinity of $\mathbf{R}P^2$ on their boundary increases from $k+1$ to $2k$. Hence, to each arc of $\mathcal{A}$ is associated a critical of index 1.

-Let us study the set $\mathcal{P}$ of critical point 1 associated to the set $\mathcal{A}$ of arcs of $\mathbf{R}\mathcal{H}_{2k}$.
Recall that we consider on $\mathbf{R}P^2$ the Fubini-Study metric induced by the projection $\pi_{\mathbf{R}} : S^2 \to \mathbf{R}P^2$.

Without loss of generality, one can assume that critical points of B_{2k} and points of the intersection of $\mathcal{H}_{2k}$ with L do not belong to the line $x_2 = 0$. On such assumption, consider the function $b_{2k}(0, x_1/x_2, 1) = b_{2k}(y)$

$$\frac{\partial}{\partial x_1} b_{2k}(0, x_1/x_2, 1) = \frac{\partial}{\partial y} b_{2k}(y) \frac{1}{x_2}$$
$$\frac{\partial}{\partial x_2} b_{2k}(0, x_1/x_2, 1) = \frac{\partial}{\partial y} b_{2k}(y) \frac{x_1}{-(x_2)^2}$$

Critical points of the function $b_{2k}(0, x_1/x_2, 1)$ may be defined from critical points $b_{2k}(y)$. (Obviously, for any such critical point of $b_{2k}(0, x_1/x_2, 1)$ the following equality is verified $(0 : x_1/x_2 : 1) = (0 : x_1 : x_2) = (0 : -x_1 : -x_2)$.) The function $b_{2k}(y)$ has exactly $2k$ zeroes which coincide with intersection points of $\mathcal{H}_{2k}$ with the line L. Hence, by Rolle's Theorem $2k-1$ extrema at which $b'_{2k}(y)$ must change from positive to negative. By continuity of b_{2k} and b'_{2k}, it follows the alternation of sign of $b'_{2k}(x_0, x_1, x_2)$ in an ϵ-tubular neighborhood $\mathcal{M}_\epsilon$ of the line $x_0 = 0$ in $\mathbf{C}P^2$ and thereby the existence of a set of $2k-1$ critical points of B_{2k} (i.e critical points of the affine polynomial $b_{2k}(x_1/x_0, x_2/x_0)$) in $\mathcal{M}_\epsilon$. According to Petrovskii's theory, to describe critical points of $b_{2k}(x_1/x_0, x_2/x_0)$ by means of the pencil $x_0^{2k}.(b_{2k}(x_1/x_0, x_2/x_0) - c)$ one can assume that B_{2k} is a regular polynomial. (i.e none of its critical points belongs to $x_0 = 0$, any two of them have different critical values.) Up to slightly modify coefficients of b_{2k}, any critical point $y = (0, x_1/x_2, 1)$ of $b_{2k}(y)$ gives rise to one critical point which may be chosen among two points $(\epsilon : \pm x_{0,1} : \pm x_{0,2}) \in \mathbf{R}P^2$.

Let us prove in Lemma 2.5 of Chapter 2 that without loss of generality one can assume that the $2k-1$ points of $\mathcal{P}$ belong to a line L_{ϵ_1} of $\mathcal{M}_\epsilon$ the ϵ-tubular neighborhood of $L := x_0 = 0$ where $\epsilon > 0$ is arbitrarily small.

LEMMA 2.5. *Let L be the real projective line at infinity. Denote by $\mathcal{M}_\epsilon$ the ϵ-tubular neighborhood of L in $\mathbf{R}P^2$.*

Given B_{2k} a Harnack polynomial of type $\mathcal{H}^0$ (relatively to L). There exists a regular modification of B_{2k} such that:

(1) *the modified polynomial is of type $\mathcal{H}^0$ (relatively to L).*
(2) *the set $\mathcal{P}$ is a set of $(2k-1)$ points of a line $L_{\epsilon_1} \subset \mathcal{M}_\epsilon$ and ϵ is arbitrarily small.*

proof:

Recall that any point $p \in \mathcal{P}$ is a critical point of index 1 associated to one of the $2k-1$ arcs of the set $\mathcal{A}$ (-the set of arcs of $\mathbf{R}\mathcal{H}_{2k}$ which delimit a region which contains a segment of the line at infinity on its boundary and does not contain an oval of $\mathbf{R}\mathcal{H}_{2k}$-)

-It is easy to get that we can regularly modify B_{2k} such that any $p \in \mathcal{P}$ belongs to $\mathcal{M}_\epsilon$ where $\epsilon > 0$ is arbitrarily small.

-Assume that any point $p \in \mathcal{P}$ belongs to $\mathcal{M}_\epsilon$ the ϵ-tubular neighborhood of L with ϵ is arbitrarily small. The polynomial $B_{2k}(x_0, x_1, x_2)$ is of type $\mathcal{H}^0$ relatively to the line L.
Let us now prove that one can regularly modify B_{2k} such that any $p \in \mathcal{P}$ belongs to a real projective line $L_{\epsilon_1} \subset \mathcal{M}_\epsilon$.
Let L_{ϵ_1} be a real projective line of $\mathcal{M}_\epsilon$. Denote by $x'_0 = x_0 + \alpha.x_1 + \beta.x_2$ the polynomial of L_{ϵ_1}. Then, consider the linear change of projective coordinates mapping $(x_0 : x_1 : x_2)$ to $(x'_0 : x_1 : x_2)$. Such transformation carries $B_{2k}(x_0, x_1, x_2) = x_0.B_{2k-1}(x_0, x_1, x_2) + C_{2k}(x_1, x_2)$ to
$B'_{2k}(x'_0, x_1, x_2) = x'_0.B'_{2k-1}(x'_0, x_1, x_2) + C'_{2k}(x_1, x_2)$ where
$B'_{2k-1}(x'_0, x_1, x_2) = x'_0.B'_{2k-2}(x'_0, x_1, x_2) + C'_{2k-1}(x_1, x_2)$. We shall prove that, up to regular modification of B'_{2k} and thus of B_{2k}, one can choose $C'_{2k-1}(x_1, x_2)$ in such a way that the $2k-1$ points of $\mathcal{P}$ belong to L_{ϵ_1}. Using the linear change of projective coordinates mapping $(x'_0 : x_1 : x_2)$ to $(x_0 : x_1 : x_2)$, we shall get the polynomial $B_{2k}(x_0, x_1, x_2)$.
Let us detail this construction.
Given $B_{2k}(x_0, x_1, x_2)$ of type $\mathcal{H}^0$, the polynomial $C_{2k}(x_1, x_2)$ such that $B_{2k}(x_0, x_1, x_2) = x_0.B_{2k-1}(x_0, x_1, x_2) + C_{2k}(x_1, x_2)$ has $2k$ distinct roots on the line at infinity $L := x_0 = 0$. Thus, $C_{2k}(x_1, x_2) = \Sigma_{i=0}^{2k} a_i x_1^{2k-i} x_2^i$ with $a_i \neq 0$.

It follows from the previous local study around $p \in \mathcal{P}$ that, up to regular modification, one can assume the polynomial B_{2k} (of type $\mathcal{H}^0$ relatively to L) such that any point p of $\mathcal{P}$ belongs to the ϵ-tubular neighborhood $\mathcal{M}_\epsilon$ of L where $\epsilon > 0$ is arbitrarily small. Denote by $B_{2k,0}$ such a polynomial and by $\mathcal{P}_0$ its respective set $\mathcal{P}$ of critical points.

Set $x_0 = \epsilon > 0$.
$B_{2k,0}(\epsilon, x_1, x_2) = \epsilon.B_{2k-1,0}(\epsilon, x_1, x_2) + C_{2k,0}(x_1, x_2)$
$= \epsilon.(\epsilon.B_{2k-2,0}(\epsilon, x_1, x_2) + C_{2k-1,0}(x_1, x_2)) + C_{2k,0}(x_1, x_2)$ with $B_i = x_0^i.b_i$ where b_i is the truncation of $b_{2k,0} = B_{2k,0}(1, x_1, x_2)$ monomials of $x_1^a.x_2^b$ of degree $a + b \leq i$.
Local coordinates defined in a neighborhood $U(p)$ of a point p depend principally on the first derivative and the second derivative of the function B_{2k} around p. Hence, to describe the curve in neighborhood of points $\mathcal{P}$, it is sufficient to consider the truncation $D_{2k,0}$ of $B_{2k,0}$ on monomials $x_1^a.x_2^b$ of degree $2k - 2 \leq a + b \leq 2k$. Therefore, to define a regular modification $B_{2k,t}$, $t \in [0, 1]$, of $B_{2k,0}$ of type $\mathcal{H}^0$ with $\mathcal{P}_0 \subset \mathcal{M}_\epsilon$ to $B_{2k,1}$ of type $\mathcal{H}^0$ with $\mathcal{P} \in L_{\epsilon_1}$ it sufficient to consider the truncation $D_{2k,0}$ of $B_{2k,0}$.

Without loss of generality, one can assume that the $2k-1$ points $p \in \mathcal{P}_0$ do not belong to $x_2 = 0$. In this way, for any $p \in \mathcal{P}_0$, $\frac{\partial B_{2k,0}}{\partial x_0}(p) = 0$ and $\frac{\partial B_{2k,0}}{\partial x_1}(p) = 0$. Moreover, $B_{2k,0}$ has $2k$ roots on the line $x_0 = 0$ and none of these points is a point of $\mathcal{P}$. Therefore, it follows that the set $\mathcal{P}_0 \subset \mathcal{M}_\epsilon$ may be chosen such that the truncation $D_{2k,0}(x_0, x_1, x_2)$ of $B_{2k,0}(x_0, x_1, x_2)$ is of the form $D_{2k,0} = \Sigma_{2k-2 \leq i+j \leq 2k} a_{i,j} x_0^{2k-(i+j)} x_1^i x_2^j$ with $a_{i,j} \neq 0$.
Set $B'_{2k} = B'_{2k,0} = x'_0.B'_{2k-1} + C'_{2k}$. We shall prove that there exists a regular modification of $B'_{2k,0}$ (that is of $B_{2k,0}$) which pushes any point $p \in \mathcal{P}_0$ to a point of the set $\tilde{\mathcal{P}}_0 := B'_{2k-1,1} = 0 \cap L_{\epsilon_1}$.

In such a way, any point $p \in \tilde{\mathcal{P}}_0$ verifies

$$\frac{\partial B'_{2k,1}}{\partial x'_0}(p) = 0 \tag{9}$$

Without loss of generality, on can assume that any point $p \in \tilde{\mathcal{P}}_0$ does not belong to $x_2 = 0$.

Thus, any point $p \in \tilde{\mathcal{P}}_0$ also verifies:

$$\frac{\partial B'_{2k,1}}{\partial x_1}(p) = 0 \tag{10}$$

Let us prove that the modification $B_{2k,t}$ $t \in [0,1]$, $B_{2k,0} = B_{2k}$ such that $B_{2k,1}$ verifies properties (9) and (10) is regular. We shall proceed by induction.
Denote p_i, $1 \leq i \leq 2k-1$ the set of $2k-1$ points of $\mathcal{P}_0$. Denote $L_0^{i,j}$ the unique line such that $p_i, p_j \in L_0^{i,j}$ Setting $L_0^{i,j} := x'_0 = 0$, properties (9) and (10) are verified for p_i, p_j. Choose p_1, p_2 two points of $\mathcal{P}_0$ associated to respectively γ_1 γ_2 arcs of $\mathcal{A}$ which intersect each other in one point of the line at infinity L.

Let $p_3 \in \mathcal{P}_0$ associated to $\gamma_3 \in \mathcal{A}$, such that γ_3 and γ_2 intersect each other in one point.

Setting $L_{\epsilon_1} = L_0^{1,2} := x'_0 = 0$, $L_0^{2,3} := x"_0 = x'_0 + \alpha'.x_1 + \beta'.x_2 = 0$. The linear change of projective coordinates mapping $(x'_0 : x_1 : x_2)$ to $(x"_0 : x_1 : x_2)$ carries $B'_{2k}(x'_0, x_1, x_2) = x'_0.B_{2k-1}(x'_0, x_1, x_2) + C_{2k}(x_1, x_2)$ to
$B"_{2k}(x"_0, x_1, x_2) = x"_0.B"_{2k-1}(x"_0, x_1, x_2) + C"_{2k}(x_1, x_2)$
Moreover, $\frac{\partial B"_{2k,0}}{\partial x"_0}(p_3) = 0$, $\frac{\partial B"_{2k,0}}{\partial x_1}(p_3) = 0$
One can modify regularly coefficients of $B_{2k,0}$ such that $p_3 \in L_{\epsilon_1} = L_0^{1,2}$. (i.e there exists regular modification on gradient trajectories such that p_3 is pushed to a point of the line $L_{\epsilon_1} = L_0^{1,2}$) This modification may be expressed from the path $x"_{0,t} = x'_0 + (\alpha' - t.\alpha').x_1 + (\beta' - t.\beta').x_2$, $t \in [0,1]$ from $L_0^{2,3}$ to L_{ϵ_1} and polynomials $B"_{2k-1,t}(x"_{0,t}, x_1, x_2)$ and $C"_{2k,t}(x_1, x_2)$ $B"_{2k,t} = x"_{0,t}.B"_{2k-1,t}(x"_{0,t}, x_1, x_2) + C"_{2k,t}(x_1, x_2)$. The deformation moves $p_{3,t} \in L_t$ in such a way that $\frac{\partial B"_{2k,t}}{\partial x"_{0,t}}(p_{3,t})$ and $\frac{\partial B"_{2k,t}}{\partial x_1}(p_{3,t})$. Note that $B"_{2k,0} = B"_{2k}, p_{3,0} = p_3 \in L_0^{2,3}$, $p_{3,t} \in L_t$ where $x"_{0,t}$ is the polynomial of L_t and $L_1 = L_{\epsilon_1}$) $(B"_{2k,1} = x"_{0,1}.B"_{2k-1,1}(x"_{0,1}, x_1, x_2) + C"_{2k,1}(x_1, x_2)$; $B"_{2k,1} = x'_0.B"_{2k-1,1}(x'_0, x_1, x_2) + C'_{2k,1}(x_1, x_2) \neq x'_0.B_{2k-1}(x'_0, x_1, x_2) + C_{2k}(x_1, x_2)$.) Since $p_1, p_2, p_3 \in \mathcal{M}_\epsilon$, where $\epsilon > 0$ is arbitrarily small; $|\alpha'|$ and $|\beta'|$ are also arbitrarily small. In this way, (see Lemma 1 and Lemma 2 of [**16**]), the deformation $B"_{2k,t} = x"_{0,t}.B"_{2k-1,t}(x"_{0,t}, x_1, x_2) + C"_{2k,t}(x_1, x_2)$ such that $p_{3,t}$ is a root of $B"_{2k-1,t}(0, x_1, x_2)$ (i.e of $\frac{\partial B"_{2k,t}}{\partial x"_{0,t}}$) and of $\frac{\partial B"_{2k,t}}{\partial x_1}$, $t \in [0,1]$ is regular.

Iterating this argument for p_i, $4 \leq i \leq 2k-1$, we get the announced regular deformation $B_{2k,t}$, $t \in [0,1]$, of B_{2k}. Any $p_i \in \mathcal{P}_0$ is pushed to a root of $C'_{2k-1,1}(x_1,1)$ (that is a root of $\frac{\partial B'_{2k,1}}{\partial x'_0}$) which is also a root of $\frac{\partial C'_{2k,1}}{\partial x_1}(x_1,1)$ (that is a root of $\frac{\partial B'_{2k,1}}{\partial x_1}$)

For any p_i, $1 \leq i \leq 2k-1$, of $\tilde{\mathcal{P}}_0 := B'_{2k-1,1} = 0 \cap L_{\epsilon_1}$ consider its representative $(0 : x_i : 1)$.

Let s_j, $j \in \{1,...,2k-1\}$ be the elementary roots symmetric polynomial of degree j on the p_i.

The polynomial $C'_{2k,1}(x_1,x_2)$ of $B'_{2k,1}(x'_0,x_1,x_2) = x'_0.B'_{2k-1,1}(x'_0,x_1,x_2) + C'_{2k,1}(x_1,x_2)$ is (up to multiplication by a constant) such that:

$$\frac{\partial B'_{2k,1}}{\partial x_1}(0,x_1,x_2) = x_1^{2k-1} - s_1 x_1^{2k-2}x_2 + ...(-1)^{2k-1}s_{2k-1}x_2^{2k-1}$$

Bringing together the resulting equalities, it follows the polynomial $B'_{2k,1}(x'_0,x_1,x_2)$ and thus the definition of $B_{2k,1}(x_0,x_1,x_2)$ with $\mathcal{P} = \tilde{\mathcal{P}}_0 \subset L_{\epsilon_1}$. Since $L_{\epsilon_1} \subset \mathcal{M}_\epsilon$, with $\epsilon > 0$ arbitrarily small, the modification $B_{2k,t}$, $t \in [0,1]$, of $B_{2k,0}$ of type $\mathcal{H}^0$ with $\mathcal{P}_0 \subset \mathcal{M}_\epsilon$ is such that $B_{2k,1}$ with $\mathcal{P}_1 \in L_{\epsilon_1}$ is also of type $\mathcal{H}^0$.

Remark 2.6. Note that the rigid proposed above requires a balance of critical value. Hence, it does not push arcs $\mathcal{A}$ and $\mathcal{A}'$ to $\mathcal{M}_\epsilon$. Indeed, consider the Fubini-Study metric on S^2 and in such a way consider $\mathbf{R}\mathcal{H}_{2k} \subset S^2$. Any point $p \in \mathcal{P}$ has representative in S^2 $(x_0,x_1,x_2) = (\epsilon,p_1,p_2)$ with $\epsilon << 1$.

$$B_{2k}(x_0,x_1,x_2) = x_0^{2k-2}B_{2k-2}(x_0,x_1,x_2) + D_{2k}$$

Hence, since $\epsilon^{2k-2}.||B_{2k-2}|| << 1$ for $\epsilon << 1$; In a neighborhood of p,

$$B_{2k}(\epsilon,p_1,p_2) = \epsilon^{2k-2}B_{2k-2}(\epsilon,p_1,p_2) + D_{2k}(\epsilon,p_1,p_2) \approx D_{2k}(\epsilon,p_1,p_2)$$

It follows that the "depth of waves" $\mathcal{A} \cup \mathcal{A}'$ depends principally on coefficients of monomials $x_1^{2k-i}x_2^i$ of C_{2k}; $B_{2k} = x_0.B_{2k-1} + C_{2k}$)

It concludes the proof of Lemma 2.5 of Chapter 2. Q.E.D

-Let us describe the set $\mathcal{A}'$ associated to $\mathcal{P}$ and $\mathcal{A}$. To this end, we shall study the pencil $x_0^{2k}.(b_{2k}(x_0,x_1,x_2) - c)$ $c \in \mathbf{R}$.

Let us recall, for sake of clarity, some properties of this pencil (see Lemma 1.3 of Chapter 2).

Let c_M be the maximal critical value, let c_m be the minimal critical value. Denote by $M_c = \{x \in \mathbf{R}^2 | b_{2k} > 0\}$ For $c > c_M$, M_c has $2k$ connected components. For $c < c_m$, M_c is connected.
Two components of the connected components can coalescence only when b_{2k} passes a critical value 1. Bringing together this statement with Lemma 1.2 of Chapter 2, and the fact that any curve of the pencil over $\mathcal{H}_{2k}$ intersects the line $x_0 = 0$ in $2k$ points, it follows easily that an arc of $\mathcal{A}$ bounding a positive region may not touch an arc of $\mathcal{A}$ bounding a negative region as c varies from 0.

Let us prove the following Lemma.

LEMMA 2.7. *As c varies from 0 to $+\infty$ or $-\infty$ any two arcs γ, γ' of $\mathcal{A}$ may not be deformed in such a way that for $c = c_0$ they touch each other.*

proof: Let $\mathcal{H}_{2k}$ be a curve of type $\mathcal{H}^0$ relatively to L and B_{2k} its polynomial. The proof of Lemma 2.7 of Chapter 2 is based on a study of $\mathbf{R}\mathcal{H}_{2k}$ on a tubular neighborhood $\mathcal{M}_\epsilon$ of $\mathbf{R}L$ and in particular the two following facts.
According to Lemma 2.5 of Chapter 2, given a curve $\mathcal{H}_{2k}$ of type $\mathcal{H}^0$ relatively to L, one can push by a rigid isotopy the set $\mathcal{P}$ associated to $\mathcal{A}$ to a line of the tubular neighborhood $\mathcal{M}_\epsilon$ of $\mathbf{R}L$ where ϵ positive is arbitrarily small.
According to the Petrovskii's theory, [**16**], up to slightly modify coefficients of B_{2k} without changing topological structure of $\mathcal{H}_{2k}$ one can assume that none of its critical point belongs to the line at infinity.

Choose $\mathcal{M}_\epsilon$ the tubular neighborhood of $\mathbf{R}L$ such that any line of$\mathcal{M}_\epsilon$ intersects the non-empty oval of $\mathcal{H}_{2k}$ in $2k$ points. According to proof of Lemma 2.5 of Chapter 2, one can assume that critical points of B_{2k} of the set $\mathcal{P}$ belong to L_{ϵ_1} where $\mathbf{R}L_{\epsilon_1} \subset \mathcal{M}_\epsilon$ and $\epsilon > 0$ is arbitrarily small. In this way, as c increases two arcs γ, γ' of $\mathcal{A}$ bounding positive regions are deformed and for $c = c_0$ they touch each other in a point of the line $\mathbf{R}L_{\epsilon_1} \subset \mathcal{M}_\epsilon$. The connected set $\mathcal{A}$ is such that these two arcs are separated from each other by an arc of $\mathcal{A}$ which bounds a negative region. As c increases from c_0 to $c_0 + \epsilon$, a negative oval $\mathcal{O}$ appears. The oval $\mathcal{O}$ does not intersect $\mathbf{R}L_{\epsilon_1}$, it intersects the line at infinity $\mathbf{R}L$ in two points. Up to slightly modify coefficients of B_{2k} without changing topological structure of $\mathcal{H}_{2k}$ one can consider $\mathbf{R}L_{\epsilon_1}$ as the line at infinity of $\mathbf{R}P^2$. The creation of this negative oval leads to contradiction with the number of critical points of B_{2k} and also with Petrovskii's Lemma 1.3 of Chapter 2. Indeed, as c grows, this negative oval shrinks in point and then disappears; the number of intersection points of $\mathcal{H}_{2k}$ with the line L_{ϵ_1} decreases from $2k$ to $2k - 2$.
An argumentation analogous to the previous one may be used to prove that as c increases from 0 to $+\infty$, two arcs bounding a negative region may not be deformed and touch each other.
Moreover, when consider the Petrovskii's pencil over b_{2k}, it follows easily that one arc bounding a positive region may not touch an arc bounding a negative region as c varies from 0.
This proves Lemma 2.7 of Chapter 2.
Q.E.D

We may describe, according to Petrovskii's Lemmas 1.2 of Chapter 2 and 1.3 of Chapter 2, critical points of index 1 associated to arcs of $\mathcal{A}$ as follows. There exist $2k - 3$ critical points with critical value $c_0 > 0$ such that for $c = c_0$, an arc γ of $\mathcal{A}$ touches an oval.-(Each of the $(k - 1)$ arcs bounding negative region touches a positive oval, the $(k - 2)$ arcs which bound positive region touch negative ovals)-
There exist 2 arcs of $\mathcal{A}$ bounding a positive region $\{x \in \mathbf{R}^2 | b_{2k}(x) > 0\}$ for which there exists $c_0 < 0$ with the following property. As c decreases from 0 to c_0, the region expands and for $c = c_0$ the arc touches a positive oval. (Note that, at this time of our proof, any positive oval under consideration above may be the non-empty one). In such a way, according to Morse Lemma since any point $p \in \mathcal{P}$ is a critical point of index 1 of $B_{2k}(x_0, x_1, x_2)$, any arc γ of $\mathcal{A}$ is associated to an arc γ' of $\mathbf{R}\mathcal{H}_{2k}$. We shall denote by $\mathcal{A}'$ the set of arcs γ'.

Denote by $\mathcal{M}$ a tubular neighborhood of the line at infinity. As already noticed, arcs $\mathcal{A}\cap\mathbf{R}^2$ may be seen (up to isotopy) as arcs of $\mathbf{R}L_1^+$ and $\mathbf{R}L_2^+$. In the same way, arcs $\mathbf{R}\mathcal{H}_{2k}$ of $\mathcal{A}'$ may be pushed on the boundary and the inside of $\mathcal{M}$. Any oriented arc of the $k-1$ arcs of $\mathbf{R}L_2^+$ is associated (to a critical point with positive critical value) to an arc of $\mathbf{R}L_1^+$, any oriented arc of the $k-3$ arcs of $\mathbf{R}L_1^+$ associated to a critical point with positive critical value is associated to an arc of $\mathbf{R}L_2^+$. (According to Petrovskii's theory, as c increases, these arcs recede from $\mathbf{R}L$.) These arcs are associated to a critical point with positive critical value. The two other arcs of $\mathbf{R}L_1^+$ are associated to a critical point with negative critical value and to an arc lying in the inside of $\mathcal{M}^-$. These two arcs contain an extremity of $\mathbf{R}L^+$. (Let $(\gamma,\gamma')\in(\mathcal{A},\mathcal{A}')$ be such a pair of associated arcs. According to Petrovskii's theory, as c decreases, the arc γ comes closer to $\mathbf{R}L\subset\mathcal{M}$, the arc γ' comes closer to an extremity of $\mathbf{R}L^+$.)

-Let us extend this description of $\mathbf{R}\mathcal{H}_{2k}$ to a description in the whole $\mathcal{M}$.

Let $\mathcal{A}$ be a curve and A its polynomial. We shall say that the truncation D of A is *sufficient* for $\mathcal{C}\subset\mathcal{A}$ in $\mathbf{R}P^2$ if the curve $\mathcal{D}$ with polynomial D is such that : $\mathbf{R}\mathcal{D}$ is embedded in $\mathbf{R}\mathcal{A}$ and $\mathbf{R}\mathcal{D}$ is homeomorphic to a subset of $\mathbf{R}\mathcal{A}$ which contains $\mathcal{C}$.

Let us prove the following Lemma:

LEMMA 2.8. *Let $B_{2k}(x_0,x_1,x_2)$ be a Harnack polynomial of degree $2k$ and type $\mathcal{H}^0$. Up to regular modification of $B_{2k}(x_0,x_1,x_2)$,*

(1) *the truncation of $b_{2k}^{\Delta}(x,y)$ on monomials of homogeneous degree $2k-2\le i\le 2k$ is sufficient for $\mathcal{A}\cup\mathcal{A}'$.*
(2) *there exists a truncation (on four monomials $x^cy^d, x^{c+1}y^d, x^cy^{d+1}, x^{c+1}y^{d+1}$ with $c+d=2k-2$) B_{2k}^S of B_{2k}^{Δ} for which B_{2k}^S is sufficient for $\gamma\cup\gamma'$ where $\gamma\in\mathcal{A}$ $\gamma'\in\mathcal{A}'$.*

Denote by $\mathcal{H}_{2k}^{\Delta}$ the curve with polynomial B_{2k}^{Δ}. -Roughly speaking, it means that monomials which are not in b_{2k}^{Δ} (resp, b_{2k}^{S}) have a small influence on $\mathcal{A}\cup\mathcal{A}'\subset\mathbf{R}\mathcal{H}_{2k}$ (resp, $\gamma\cup\gamma'\subset\mathbf{R}\mathcal{H}_{2k}$)-

proof: The proof of Lemma 2.8 of Chapter 2 is based on Lemma 2.5 of Chapter 2.

According to the proof of Morse Lemma, local coordinates defined in a neighborhood $U(p)$ of a non-degenerate critical point p of a function f depend principally on the first derivative and the second derivative of the function around this point. Let L_{ϵ_1} be a real projective line in the ϵ-tubular neighborhood of $x_0=0$, where $\epsilon>0$ is arbitrarily small. On the assumption that any critical point p belongs to the line L_{ϵ_1}, monomials which are not in b_{2k}^{Δ} have a small influence on $\{(x_0,x_1,x_2)\in\mathbf{R}P^2|B_{2k}(x_0,x_1,x_2)=0\}\cap U(p)$. Without loss of generality, we may assume that points at infinity of B_{2k} and points of $\mathcal{P}$ do not belong $x_2=0$. In such a way, any point at infinity of an arc of $\mathcal{A}$ is a root of the polynomial in one variable $b_{2k}(0,x_1/x_2,1)=b_{2k}(y)$. Any arc of $\mathcal{A}$ intersects the line $x_0=0$ in two points. Up to regular modification the $2k-1$ points of $\mathcal{P}$ belong to L_{ϵ_1}. According to Rolle's Theorem, $b_{2k}(y)$ has exactly $2k-1$ extrema at which $b'_{2k}(y)$ must change from positive to negative. Thus, $b_{2k}"(y)$ has $2k-2$ zeroes. By continuity, it gives the sign of the function $b_{2k}(x_0,x_1,x_2)$ in a neighborhood $\mathcal{M}=\{(x_0:x_1:x_2)\in\mathbf{R}P^2||x_0|<\epsilon\}$ of $x_0=0$. In a neighborhood $U(p)$ of an extremum, $\{x\in\mathbf{R}P^2|b_{2k}=0\}\cap U(p)$ is described as a desingularized crossing. From the previous study, it follows that up

to regular modification of B_{2k}, properties 1. and 2. are verified. One can also notice that the alternation of sign of B_{2k} in the inside of $\mathcal{M}$ is equivalent to the modular property of the distribution of sign of b_{2k}^{Δ} in the patchworking construction. Set $\mathcal{N}$ tubular neighborhood of L in $\mathbf{C}P^2$ with real part $\mathcal{M}$. It is easy to verify that replacing $\mathcal{N}$ in place of $\mathcal{M}$ one get a general formulation of the Lemma 2.8 of Chapter 2 in $\mathbf{C}P^2$ This last remark concludes and completes our proof. Q.E.D

According to the Lemma 2.8 of Chapter 2 the previous description of $\mathbf{R}\mathcal{H}_{2k}$ in $U(p)$ enlarges to a description of $\mathbf{R}\mathcal{H}_{2k}$ in $U \supset U(p)$ where

$$U = \{z =< u, p >= (u_0.p_0 : u_1.p_1 : u_2.p_2) \in \mathbf{C}P^2 | u = (u_0 : u_1 : u_2) \in U^3_{\mathbf{C}}, p = (p_0 : p_1 : p_2) \in U(p)\}$$

Local coordinates y_1, y_2 in $U(p)$ extend to local coordinates in U as follows. Given $z =< u, p >\in U$ with $u = (1 : u_1 : u_2) \in U^3_{\mathbf{C}}$, $p = (p_0 : p_1 : p_2) \in U(p)\}$, we may set $y_1(z) = u_1.y_1(p)$, $y_2(z) = u_2.y_2(p)$. In $U(p)$, the truncation b^S_{2k} is ϵ-sufficient for b_{2k}.
$b^S_{2k}(x, y) = l(x, y) + \epsilon.k(x, y)$
with $l(x, y) = a_{c,d}x^c y^d + a_{c+1,d}x^{c+1}y^d$, $k(x, y) = a_{c,d+1}x^c y^{d+1} + -a_{c+1,d+1}x^{c+1}y^{d+1}$, (with $a_{c,d} > 0, a_{c+1,d} > 0, a_{c,d+1} > 0, a_{c+1,d+1} > 0$ and $c + d = 2k - 2$ and $\epsilon > 0$)

Note that up to modify the coefficients $a_{c,d}, a_{c,d+1}, a_{c+1,d}, a_{c+1,d+1}$ if necessary, the point $p = (x_0, y_0)$ is (up to homeomorphism) a critical point of the function $\frac{l(x,y)}{k(x,y)}$ with positive critical value. Hence, it follows from the equalities
$l(x, -y) = -l(x, y)$, $k(x, -y) = k(x, y)$,
$\frac{\partial l}{\partial x}(x, -y) = -\frac{\partial l}{\partial x}(x, y)$, $\frac{\partial l}{\partial y}(x, -y) = \frac{\partial l}{\partial y}(x, y)$,
$\frac{\partial k}{\partial x}(x, -y) = \frac{\partial k}{\partial x}(x, y)$, $\frac{\partial k}{\partial y}(x, -y) = -\frac{\partial k}{\partial y}(x, y)$
that $(x_0, -y_0)$ is also a critical point of the function $\frac{l(x,y)}{k(x,y)}$ with negative critical value.

The tubular neighborhood of $\mathbf{R}L^+$ is $\mathcal{M}^+ = \cup_{p\in\mathcal{P}} U(p)$. The transformation $(x, y) \to (-x, y)$ defined locally inside any $U(p)$, $p \in \mathcal{P}$ maps the set of arcs $\mathcal{A} \cup \mathcal{A}'$ to a set of arcs $\mathcal{B}$ and the set of points $\mathcal{P}$ to a set of points $\mathcal{S}$. Any open $U(p)$ is mapped to an open $U(s)$ in such a way that the tubular neighborhood of L^- is $\mathcal{M}^- = \cup_{s\in\mathcal{S}} U(s)$ and L^+ is mapped to L^-.

Precisely, given x, y local coordinates in a neighborhood $U(p)$, the pair $(\gamma, \gamma') \in (\mathcal{A}, \mathcal{A}')$ of $\{(x_0 : x_1 : x_2) \in \mathbf{R}P^2 | B_{2k}(x_0, x_1, x_2) = 0\} \cap U(p)$ is up to homeomorphism the hyperbole $x.y = 1/2$. By means of the transformation $(x, y) \to (x, -y)$ one maps $U(p)$ onto $U(s)$ and the pair of arcs $(\gamma, \gamma') \in (\mathcal{A}, \mathcal{A}')$ defined up to homeomorphism by $x.y = 1/2$ to the pair of arcs (ξ, ξ') of $\{(x_0 : x_1 : x_2) \in \mathbf{R}P^2 | B_{2k}(x_0, x_1, x_2) = 0\} \cap U(p)$ is up homeomorphism the hyperbole $x.y = -1/2$. (In $U(s)$, one can say that arcs ξ,ξ' are *associated* and also associated to s.)

Let us in Lemma 2.9 of Chapter 2 describe the set $\mathcal{D} = \mathcal{M}^- \cap \mathbf{R}\mathcal{H}^{\Delta}_{2k}$.

LEMMA 2.9. *Let $\mathcal{H}_{2k}$ be a Harnack curve of type $\mathcal{H}^0$ with polynomial B_{2k}. Up to regular modification of the Harnack polynomial B_{2k}, $\mathbf{R}\mathcal{H}^{\Delta}_{2k}$ intersects $\mathcal{M}^-$ in $2k - 2$ positive ovals.*

proof:

The set $\mathcal{A}$ is a connected set of arcs which intersects $\mathbf{R}L$ the line at infinity of $\mathbf{R}P^2$ in $2k$ points which are intersection points of two arcs of $\mathcal{A}$. Using Morse Theory, one can consider these points as limit points of an half hyperbole. Up to isotopy of $\mathbf{R}P^2$, one can consider arcs of $\mathcal{A} \cup \mathcal{A}'$ as arcs lying on $\mathbf{R}L_1^+$ and $\mathbf{R}L_2^+$.

Except the two arcs pushed by an isotopy of $\mathbf{R}P^2$ to a segment of $\mathbf{R}L_1^+$ which contains an extremity of $\mathbf{R}L_1^+$, each arc of $\mathcal{A}$ is associated to an arc of $\mathcal{A}'$ in such a way that the set $\mathcal{A} \cup \mathcal{A}'$ is the union of two connected parts lying respectively on $\mathbf{R}L_1^+$ and $\mathbf{R}L_2^+$.

Each of the two arcs which contains an extremity of $\mathbf{R}L_1^+$ is associated to an arc $\mathcal{A}'$ which belongs to the part $\mathcal{M}^-$ of the Möbius band lying between $\mathbf{R}L_1^-$ and $\mathbf{R}L_2^-$. In our identification of arcs with connected real parts lying in the inside of $\mathcal{M}$, one can assume that these two pairs remain the same under the action $(x, y) \to (x, -y)$ which maps arcs $\mathcal{A} \cup \mathcal{A}'$ to arcs of $\mathcal{B}$. (Indeed, $\mathcal{P}$ intersects $\mathcal{S}$ in two points which are extremities of $\mathbf{R}L_{\epsilon_1}^+ \approx \mathbf{R}L$.)

It follows that the union of $(\mathcal{A} \cup \mathcal{A}')$ is mapped to $2(k-1)$ ovals constituted by the union of arcs $\mathcal{B}$ union two arcs of $\mathcal{M}^+$ which contains an extremity of $\mathbf{R}L_1^+$.

These ovals lie in the inside of the Möbius band $\mathcal{M}^-$ and intersects $\mathbf{R}L^-$ in two points. This last remark concludes our proof. Q.E.D

Remark 2.10. It follows from the Lemma 2.9 of Chapter 2 a description, up to isotopy of $\mathbf{R}P^2$, of $\mathbf{R}\mathcal{H}_{2k}$ in $\mathcal{M}$ which extends our previous description of $\mathcal{A} \cup \mathcal{A}' \subset \mathbf{R}\mathcal{H}_{2k}$ Note that this description is consistent with the study of the one variable function $b_{2k}(y) = b_{2k}(0, x_1/x_2, 1)$. The function $b_{2k}"(y)$ has $2k-2$ zeroes. Any point at infinity of an arc of $\mathcal{A}$ is a root of the polynomial in one variable $b_{2k}(0, x_1/x_2, 1) = b_{2k}(y)$. According to Rolle's Theorem, $b_{2k}(y)$ has exactly $2k-1$ extrema at which $b'_{2k}(y)$ must change from positive to negative. The zeroes of $b_{2k}"(y)$ are simple zeroes. These zeroes are the intersection-points of the set $\mathcal{B}$ with L.

The set $\mathcal{A}$ is the image of a smooth section of a tubular fibration of $\mathbf{R}L^+$ minus 2 points. In Lemma 2.11 of Chapter 2, we shall prove that there exists a subset $\mathcal{D}^1$ of $\mathcal{D} = \mathbf{R}\mathcal{H}_{2k} \cap \mathcal{M}^-$ with the property that the set $\mathcal{A} \cup \mathcal{D}^1$ is the image of a smooth section of a tubular fibration of $\mathbf{R}L$ minus a finite number of points.

By cutting $\mathcal{M}^-$ along L^-, one get two surfaces. Denote by $\mathcal{M}^{1,-}$ the one which contains L_1^-. Consider the intersection $\mathbf{R}\mathcal{H}_{2k} \cap \mathcal{M}^{1,-}$

Any arc ξ of $\mathcal{D} = \mathbf{R}\mathcal{H}_{2k} \cap \mathcal{M}$ intersects the line at infinity L. Hence, it is divided into two halves with common point $L \cap \xi$. Denote by ξ^1 (resp, ξ^2) the half of ξ which belongs to the inside of Möbius delimited by $\mathbf{R}L$ and $\mathbf{R}L_1$ (resp, $\mathbf{R}L$ and $\mathbf{R}L_2$) minus its intersection with L. The set $\mathbf{R}\mathcal{H}_{2k} \cap \mathcal{M}^{1,-}$ is the set of arcs ξ^1.

In Lemma 2.11 of Chapter 2 we prove that $\mathcal{D}^1 = \mathbf{R}\mathcal{H}_{2k} \cap \mathcal{M}^{1,-}$

Lemma 2.11. *Let $\mathcal{H}_{2k}$ be a Harnack curve of type $\mathcal{H}^0$ with polynomial B_{2k}. Up to regular modification of the Harnack polynomial B_{2k}, the set $\mathcal{A} \cup \mathcal{D}^1$, where $\mathcal{D}^1 = \mathbf{R}\mathcal{H}_{2k}^{\Delta} \cap \mathcal{M}^{1,-}$, is the image of a smooth section of a tubular fibration of $\mathbf{R}L$ minus $2k-1$ points.*

proof: The set $\mathcal{A}$ is connected. There exists an isotopy of $\mathbf{R}P^2$ which pushes the union of the k arcs of $\mathcal{A} \cap \mathbf{R}^2$ bounding positive regions $\{x \in \mathbf{R}^2 | b_{2k}(x) > 0\}$

with $k-1$ arcs of $\mathcal{A}'$ (associated to the $k-1$ arcs of $\mathcal{A}\cap\mathbf{R}^2$ bounding negative regions$\{x\in\mathbf{R}^2|b_{2k}(x)>0\}$) onto $\mathbf{R}L_1^+$. Let us denote by $\mathcal{A}^1$ this set of k arcs of $\mathcal{A}$ with $(k-1)$ arcs of $\mathcal{A}'$ arcs. The transformation $(x,y)\to(x,-y)$ maps $U(p)\to U(s)$ and the pair of arcs $(\gamma,\gamma')\in(\mathcal{A},\mathcal{A}')$ defined up to homeomorphism by $x.y=1/2$ to the pair of arcs $(\xi,\tilde{\xi})$ of $\mathcal{B}$ defined up to homeomorphism by $x.y=-1/2$. Hence, identifying each arc of $\mathcal{A}^1$, with a segment of $\mathbf{R}L_1^+\cap U(p)$, it follows the next descriptions in $U(s)$. Let us first consider an open $U(s)$ which does not contain an extremity of L. According to Morse Lemma in $U(s)$, it follows that the pair $(\xi^1,\tilde{\xi}^1)$ of halves of associated arcs in $U(s)$ is the image of a smooth section of a tubular fibration of $U(s)\cap\mathbf{R}L_1^-$ minus one point.

If $U(s)$ contains an extremity of L, then the pair of arcs $(\xi^1,\tilde{\xi}^1)=(\xi^1\cap\mathcal{M}^-,\tilde{\xi}^1\cap\mathcal{M}^+)$ of halves of associated arcs in $U(s)$ is such that ξ^1 is the image of a smooth section of a tubular fibration of $U(s)\cap\mathbf{R}L_1^-$ minus the extremity of $\mathbf{R}L_1^-\cap U(s)$. By means of the transformation $(x,y)\to(x,-y)$, inside any open $U(p)$, the tubular neighborhood $\mathcal{M}^+=\cup_{p\in\mathcal{P}}U(p)$ of L^+ is mapped to the tubular neighborhood $\mathcal{M}^-=\cup_{s\in\mathcal{S}}U(s)$ of L^-. It follows from Morse Lemma that the set $\mathcal{D}^1$ is the image of a smooth section of tubular fibration of $\mathbf{R}L_1^-$ minus $2k-1$ points.

Hence, the set $\mathcal{D}^1$ is also the image of a smooth section of tubular fibration of $\mathbf{R}L^-$ minus $2k-1$ points.

Since $\mathcal{P}$ intersects $\mathcal{S}$ in the extremities of $\mathbf{R}L_{\epsilon_1}^+$ (see proof of Lemma 2.9 of Chapter 2), two of these $2k-1$ points of $\mathbf{R}L^-$ are extremities.

Hence, $\mathcal{D}^1$ is the image of a smooth section of tubular fibration of $\mathbf{R}L^-$ minus $\mathcal{S}$. Therefore, since $\mathcal{A}$ is the image of a smooth section of tubular fibration of $\mathbf{R}L_+$ minus its extremities, $\mathcal{A}\cup\mathcal{D}^1$ is the image of a smooth section of tubular fibration of $\mathbf{R}L$ minus $2k-1$ points. Q.E.D

According to Proposition 1.4 of Chapter 2, given B_{2k} and B_{2k-1} Harnack polynomials of type $\mathcal{H}^0$ and respective degree $2k$ and $2k-1$

$$c(B_{2k})=(\frac{(k-1)(k-2)}{2},k.(2k-1),\frac{3k.(k-1)}{2}))$$

$$c(B_{2k-1})=(\frac{(k-1).(k-2)}{2},(k-1).(2k-1),\frac{(k-1).(3k-4)}{2})$$

The Harnack curve $\mathcal{H}_{2k}$ of degree $2k$ has $2(k-2)$ ($:=c_2(B_{2k})-c_2(B_{2k-1})$) more positive ovals than the Harnack curve $\mathcal{H}_{2k-1}$ of degree $2k-1$. It follows from Lemma 2.8 of Chapter 2 and Lemma 2.9 of Chapter 2 that, up to regular to modification of the Harnack polynomial B_{2k}, these ovals are the ovals of $\mathbf{R}\mathcal{H}_{2k}^{\Delta}\cap\mathcal{M}^-$.

In such a way, it follows from Lemma 2.11 of Chapter 2 that, up to regular modification of B_{2k}, $B_{2k}=x_0.B_{2k-1}+C_{2k}$ where B_{2k-1} is a Harnack polynomial of type $\mathcal{H}^0$ relatively to $L:=x_0=0$ the line at infinity. (Indeed, as we have noticed, the truncation B_{2k}^{Δ} is sufficient for B_{2k} and L_{ϵ_1} may be chosen arbitrarily close to L.) It concludes our proof of Proposition 2.4 of Chapter 2. **Q.E.D**

I.ii)-Harnack curves of odd degree-

Let $\mathcal{H}_{2k+1}$ be a Harnack curve of degree $2k+1\geq 5$ with polynomial $B_{2k+1}(x_0,x_1,x_2)$ of type $\mathcal{H}^0$ relatively to the line at infinity $L:=x_0=0$.

We shall prove the following Proposition 2.12 of Chapter 2.

Proposition 2.12. *Let $\mathcal{H}_{2k+1}$ be a Harnack curve of degree $2k+1$ with polynomial $B_{2k+1}(x_0,x_1,x_2)$ of type $\mathcal{H}^0$ relatively to the line at infinity $L := x_0 = 0$. Up to regular modification of B_{2k+1}, $B_{2k}(x_0,x_1,x_2)$ is also of type $\mathcal{H}^0$ relatively to L.*

proof:
The part I.ii) until its end is devoted to the proof of Proposition 2.12 of Chapter 2. Our argumentation is a slightly modified version of the one given in the proof of Proposition 2.12 of Chapter 2. When it is possible, we refer to the first part I.i).

The odd component of $\mathcal{H}_{2k+1}$ is divided into $2k+1$ arcs which delimit a region bounding the line at infinity. One of this region contains ovals of $\mathcal{H}_{2k+1}$. Denote $\mathcal{A}$ the set of $2k$ arcs of $\mathbf{R}\mathcal{H}_{2k+1}$ which delimit a positive region $\{x \in \mathbf{R}^2 | b_{2k+1} > 0\}$ (i.e region which does not contain oval of $\mathcal{H}_{2k+1}$). The union of arcs of $\mathcal{A}$ is a connected orientable part of the odd component of $\mathbf{R}\mathcal{H}_{2k+1}$. The odd component of $\mathbf{R}\mathcal{H}_{2k+1}$ is homeomorphic to a projective line. Consider an orientation of the line at infinity $\mathbf{R}L$. The surface delimited by the $2k$ arcs of $\mathcal{A}$ and $\mathbf{R}L$ is orientable. Thus, one can assume that the orientation of the connected union of the $2k$ arcs of $\mathcal{A}$ is induced by a half of $\mathbf{R}L$. We shall denote by $\mathbf{R}L^+$ this half. The set $\mathcal{A}$ is the image of a smooth section of a tubular fibration of $\mathbf{R}L^+$ minus 2 points.

Let us prove that one can associate to the set $\mathcal{A}$ of the $2k$ arcs a set of critical points $\mathcal{P}$ of index 1 of b_{2k+1} and a set of arcs $\mathcal{A}'$ of $\mathbf{R}\mathcal{H}_{2k+1}$.
To this end, we shall study the Petrovskii's pencil $x_0^{2k+1}.(b_{2k+1}(x_1/x_0, x_2/x_0) - c)$, $c \in \mathbf{R}$, over $\mathcal{H}_{2k+1}$.
As c decreases from 0 positive regions and positive ovals expand. As c decreases from 0 the number of regions G_i of $\{x \in \mathbf{R}^2 | b_{2k}(x) > 0\}$ which contain a segment of the line at infinity of $\mathbf{R}P^2$ on their boundary decreases from $k+1$ to 1.
As c increases from 0 negative ovals disappear; the number of regions G_i of $\{x \in \mathbf{R}^2 | b_{2k+1}(x) > 0\}$ which contain a segment of the line at infinity of $\mathbf{R}P^2$ on their boundary remains the same. Hence, to each arc of $\mathcal{A}$ is associated a critical of index 1.

-Let us study the set $\mathcal{P}$ of critical point 1 associated to the set $\mathcal{A}$ of arcs of $\mathbf{R}\mathcal{H}_{2k}$.

Let us prove the following Lemma:

Lemma 2.13. *There exist at most two arcs of the set $\mathcal{A}$ which may be deformed as c varies from 0 to $-\infty$ or $+\infty$ in such a way that for $c = c_0$ they touch each other.*

proof: The projective plane $\mathbf{R}P^2$ may be seen as a Möbius band $\mathcal{M}$ and a disc D^2 glued along their common boundary where the core of the Möbius band is the line at infinity $\mathbf{R}L$. Assume that there exists c_0, such that as c varies from 0 to c_0 an arc γ of $\mathcal{A}$ is deformed and for $c = c_0$ it touches an oval $\mathcal{O}$ of $\mathcal{H}_{2k+1}$. The arc γ bounds positive region . Hence, if c_0 exists it is negative. The inside of the oval $\mathcal{O}$ is an orientable part of $\mathbf{R}P^2$ homeomorphic to D^2. The arc γ may be pushed by an isotopy to the boundary $\mathbf{R}L_1^+$ of the Möbius band. In this way, for $c = c_0$, γ may be also pushed to an arc $\tilde{\gamma}$ of the oval $\mathcal{O}$. Since $c_0 < 0$, according to Petrovskii's Lemmas, for $c = c_0 + \epsilon$ one can trace a non-orientable branch (i.e an arc of $\mathbf{R}L$

which intersects $\mathbf{R}L^+$ and $\mathbf{R}L^-$) of $\mathbf{R}P^2$ in the region $b_{2k} < c_0 - \epsilon$ containing γ and $\tilde{\gamma}$. The set $\mathcal{A}$ may be pushed by an isotopy onto $\mathbf{R}L_1^+$. In a neighborhood of $\mathbf{R}L_1^+$, a non-oriented branch of $\mathbf{R}P^2$ may be traced only near the extremities of $\mathbf{R}L_1^+$. Hence, the only two arcs of $\mathcal{A}$ which may glue with an oval are those which intersect an extremity of $\mathbf{R}L_1^+$. Q.E.D

It follows from Petrovskii's Lemmas 1.2 of Chapter 2, 1.3 of Chapter 2 and Lemma 2.13 of Chapter 2, that there exist $2k-2$ critical points with critical value $c_0 < 0$ such that as c varies from 0 to c_0 two arcs are deformed in such a way that for $c = c_0$ they touch each other.

As already done for Harnack curves of even degree one can study critical points $\mathcal{P}$ of B_{2k+1} using the function in one variable $b_{2k+1}(y) = b_{2k+1}(0, x_1/x_2, 1)$. According to Rolle's Theorem applied to $b_{2k+1}(y) = b_{2k+1}(0, x_1/x_2, 1)$ for any pair (γ, γ') of associated arcs of $\mathcal{A}$ there exists $\gamma" \in \mathcal{A}$ such $\gamma \cap \gamma" \neq \emptyset$ $\gamma' \cap \gamma" \neq \emptyset$.

According to Lemma 2.13 of Chapter 2 and its proof, for the 2 arcs of $\mathcal{A}$ which intersect L in one of its extremities, there exists $c_0 < 0$ such that as c varies from 0 to c_0, the arc is deformed and for $c = c_0$ it touches a positive oval.

In such a way, any arc γ of $\mathcal{A}$ is associated to a critical point p and to an other arc γ'. We denote by $\mathcal{P}$ the set of points associated to $\mathcal{A}$ and by $\mathcal{A}'$ the set of arcs associated to $\mathcal{A}$. The sets $\mathcal{A}'$ and $\mathcal{A}$ consist of $2k$ arcs of $\mathcal{H}_{2k+1}$; $2k-2$ of them belong to $\mathcal{A}$ and $\mathcal{A}'$.

-Let us give a description up to rigid isotopy of $\mathbf{R}\mathcal{H}_{2k}^{\Delta}$ in the whole $\mathcal{M}$.

Let B_{2k+1} be a Harnack polynomial of degree $2k+1$ and type $\mathcal{H}^0$.

From a version of Lemma 2.5 of Chapter 2 for odd degree curves $\mathcal{H}_{2k+1}$, it follows that there exists a rigid isotopy $B_{2k+1,t}$,$t \in [0,1]$, $B_{2k+1,0} = B_{2k+1}$, such that the modified polynomial $B_{2k+1,1}$ has the following the property: the $2k-1$ critical points $\mathcal{P}$ associated to $\mathcal{A}$ belong to a line L_{ϵ_1} of $\mathcal{M}_\epsilon$ the ϵ-tubular neighborhood of L in $\mathbf{R}P^2$ with $\epsilon > 0$ arbitrarily small.
From a version of Lemma 2.8 of Chapter 2 for odd degree, the truncation $b_{2k+1}^{\Delta}(x, y)$ of $b_{2k+1}(x, y)$ on monomials of homogeneous degree $2k-1 \leq i \leq 2k+1$ is sufficient for $\mathcal{A} \cup \mathcal{A}'$.
Denote by $\mathcal{H}_{2k+1}^{\Delta}$ the curve with polynomial $B_{2k+1}^{\Delta}(x_0, x_1, x_2) = x_0^{2k+1}.b_{2k+1}^{\Delta}(x, y)$ the description of $\mathbf{R}\mathcal{H}_{2k+1}^{\Delta}$ in any neighborhood $U(p)$ $p \in \mathcal{P}$ enlarges to a description of $\mathbf{R}\mathcal{H}_{2k+1}^{\Delta}$ in $U \supset U(p)$ where

$$U = \{z =< u, p >= (u_0.p_0 : u_1.p_1 : u_2.p_2) \in \mathbf{C}P^2 | u = (u_0 : u_1 : u_2) \in U_{\mathbf{C}}^3, p = (p_0 : p_1 : p_2) \in U(p)\}$$

Consider $\mathcal{M}^+$ the tubular neighborhood of L^+, $\mathcal{M}^+ = \cup_{p \in \mathcal{P}} U(p)$ and $\mathcal{M}^-$ the tubular neighborhood of L^-.

From an argumentation analogous to the one given in the part I.i), according to Lemma 2.8 of Chapter 2 we define a set $\mathcal{B}$ of arcs of $\mathbf{R}\mathcal{H}_{2k+1}^{\Delta}$ in $\mathcal{M}^-$.

Let (x, y) be local coordinates in $U(p)$, $p \in \mathcal{P}$. The transformation, defined locally in any $U(p)$ by $(x, y) \to (-x, y)$ maps $U(p) \to U(s)$ and in this way $\mathcal{M}^+ = \cup_{p \in \mathcal{P}} U(p)$ to $\mathcal{M}^- = \cup_{s \in \mathcal{S}} U(s)$. It maps the set of arcs $\mathcal{A} \cup \mathcal{A}'$ to a set of arcs $\mathcal{B}$ and the set of points $\mathcal{P}$ to a set of points $\mathcal{S}$.

Let us in Lemma 2.14 of Chapter 2 describe the set $\mathcal{D} = \mathcal{M}^- \cap \mathbf{R}\mathcal{H}^{\Delta}_{2k+1}$.

LEMMA 2.14. *Up to regular modification of the Harnack polynomial* B_{2k+1}, $\mathbf{R}\mathcal{H}^{\Delta}_{2k+1}$ *intersects* $\mathcal{M}^-$ *in* $k-1$ *negative ovals and* k *positive ovals.*

proof:

Our argumentation is similar to the one given in the proof of Lemma 2.9 of Chapter 2.
The set $\mathcal{A}$ is a connected set of arcs which intersects L the line at infinity of $\mathbf{R}P^2$ in $2k$ points. Using Morse Theory, one can consider intersection points of the set $\mathcal{A}$ with the line L as limit points of an half hyperbole. Except the two arcs of $\mathcal{A}$ which contain an extremity of L^+, any arc of $\mathcal{A}$ is associated to another arc of $\mathcal{A}$. Given $U(p)$ with local coordinates (x, y), the transformation $(x, y) \to (x, -y)$ maps any pair of arcs of $\mathcal{A}$ to another pair of arcs $\mathbf{R}\mathcal{H}_{2k+1}$. It follows from the alternation of signs of b_{2k+1} in any $U(p)$ of $\mathcal{M}^+$, the alternation of sign b_{2k} in any $U(s)$ of $\mathcal{M}^-$ and thereby around branches of $\mathbf{R}\mathcal{H}_{2k+1}$. It follows that any two arcs of $\mathcal{A}$ associated to each other in $U(p)$ are mapped to an arc of a positive oval and an arc of a negative oval in $U(s)$. Each of the two arcs which contains an extremity of $\mathbf{R}L^+$ is associated to an arc $\mathcal{A}'$ which belongs to $\mathcal{M}^-$. One can assume that these two pairs remain the same under the action $(x, y) \to (x, -y)$ which maps arcs $\mathcal{A} \cup \mathcal{A}'$ to arcs of $\mathcal{B}$. It follows that the union of $(\mathcal{A} \cup \mathcal{A}')$ is mapped to the union $\mathcal{B}$ of two arcs of the odd component of $\mathcal{H}_{2k+1}$ which contain extremities of $\mathbf{R}L_1^+$ (these arcs lie in $\mathcal{M}^+$) and $k-1$ negative ovals and k positive ovals of $\mathcal{H}_{2k+1}$ constituted by the union of arcs $\mathcal{B}$. These ovals lie in the inside $\mathcal{M}^-$ of the Möbius band and intersect $\mathbf{R}L^-$ in two points. This last remark concludes our proof. Q.E.D

The set $\mathcal{A}$ is the image of a smooth section of a tubular fibration of $\mathbf{R}L^+$ minus 2 points. In Lemma 2.15 of Chapter 2, we shall prove that there exists a subset $\mathcal{D}^1$ of $\mathcal{D} = \mathbf{R}\mathcal{H}^{\Delta}_{2k+1} \cap \mathcal{M}^-$ with the property that the set $\mathcal{A} \cup \mathcal{D}^1$ is the image of a smooth section of a tubular fibration of $\mathbf{R}L$ minus a finite number of points.

By cutting $\mathcal{M}^-$ along L^-, one get two surfaces. Denote by $\mathcal{M}^{1,-}$ the one which contains L_1^-. Consider the intersection $\mathbf{R}\mathcal{H}^{\Delta}_{2k+1} \cap \mathcal{M}^{1,-}$

Any arc ξ of $\mathcal{D} = \mathbf{R}\mathcal{H}^{\Delta}_{2k+1} \cap \mathcal{M}$ intersects the line at infinity L. Hence, it is divided into two halves with common point $L \cap \xi$. Denote by ξ^1 (resp, ξ^2) the half of ξ which belongs to the inside of Möbius delimited by $\mathbf{R}L$ and $\mathbf{R}L_1$ (resp, $\mathbf{R}L$ and $\mathbf{R}L_2$) minus its intersection with L. The set $\mathbf{R}\mathcal{H}^{\Delta}_{2k} \cap \mathcal{M}^{1,-}$ is the set of arcs ξ^1.

In Lemma 2.11 of Chapter 2 we prove that $\mathcal{D}^1 = \mathbf{R}\mathcal{H}^{\Delta}_{2k+1} \cap \mathcal{M}^{1,-}$

LEMMA 2.15. *Up to regular modification of the Harnack polynomial* B_{2k+1}, *the set* $\mathcal{A} \cup \mathcal{D}^1$ *is the image of a smooth section of a tubular fibration of* $\mathbf{R}L$ *minus* $2k$ *points.*

proof: Our proof is analogous to the one of Lemma 2.11 of Chapter 2. The set $\mathcal{A}$ is the image of a smooth section of a tubular fibration of $\mathbf{R}L^+$ minus 2 points The transformation $(x, y) \to (x, -y)$ maps $U(p) \to U(s)$ and the pair of arcs $(\gamma, \gamma') \in (\mathcal{A}, \mathcal{A}')$ defined up to homeomorphism by $x.y = 1/2$ to the pair of arcs $(\xi, \tilde{\xi})$ of $\mathcal{B}$ defined up to homeomorphism by $x.y = -1/2$.

Hence, identifying each of arcs of $\mathcal{A}$, with a segment of $\mathbf{R}L^+ \cap U(p)$, it follows the next descriptions in $U(s)$.

Let us first consider open $U(s)$ which does not contain an extremity of L. According to Morse Lemma in $U(s)$, it follows that the pair $(\xi^1, \tilde{\xi}^1)$ of halves of associated arcs in $U(s)$ is the image of a smooth section of a tubular fibration of $U(s) \cap \mathbf{R}L_1^-$ minus one point.
If $U(s)$ contains an extremity of L, then the pair of arcs $(\xi^1, \tilde{\xi}^1) = (\xi^1 \cap \mathcal{M}^-, \tilde{\xi}^1 \cap \mathcal{M}^+)$ of halves of associated arcs in $U(s)$ is such that ξ^1 is the image of a smooth section of a tubular fibration of $U(s) \cap \mathbf{R}L_1^-$ minus the extremity of $\mathbf{R}L_1^- \cap U(s)$.

By means of the transformation $(x, y) \to (x, -y)$, inside any open $U(p)$, the tubular neighborhood $\mathcal{M}^+ = \cup_{p \in \mathcal{P}} U(p)$ of L^+ is mapped to the tubular neighborhood $\mathcal{M}^- = \cup_{s \in \mathcal{S}} U(s)$ of L^-. It follows from Morse Lemma that the set $\mathcal{D}^1$ is the image of a smooth section of tubular fibration of $\mathbf{R}L^-$ minus $2k$ points.

Two of these $2k$ points of $\mathbf{R}L^-$ are extremities of $\mathbf{R}L^+$

Hence, $\mathcal{D}^1$ is the image of a smooth section of tubular fibration of $\mathbf{R}L_-$ minus $\mathcal{S}$. Therefore, $\mathcal{A} \cup \mathcal{D}^1$ is the image of a smooth section of tubular fibration of $\mathbf{R}L$ minus the $2k$ points. Q.E.D

According to Proposition 1.4 of Chapter 2, given B_{2k+1} and B_{2k} Harnack polynomials of type $\mathcal{H}^0$ and respective degree $2k$ and $2k-1$

$$c(B_{2k+1}) = (\frac{(k).(k-1)}{2}, k.(2k+1), \frac{(k).(3k-1)}{2})$$

$$c(B_{2k}) = (\frac{(k-1)(k-2)}{2}, k.(2k-1), \frac{3k.(k-1)}{2})$$

The Harnack curve $\mathcal{H}_{2k+1}$ of degree $2k+1$ has $k-1$ negative ovals and k positive ovals more than $\mathcal{H}_{2k}$.

It follows from the version of Lemma 2.8 of Chapter 2 for odd degree curves and Lemma 2.14 of Chapter 2 that up to regular to modification of the Harnack polynomial B_{2k+1}, these ovals are the ovals of $\mathbf{R}\mathcal{H}^{\Delta}_{2k+1} \cap \mathcal{M}^-$.

In such a way, it follows from Lemma 2.15 of Chapter 2 that, up to regular modification of B_{2k+1}, $B_{2k+1} = x_0.B_{2k} + C_{2k+1}$ where B_{2k} is a Harnack polynomial of type $\mathcal{H}^0$ relatively to $x_0 = 0$ the line at infinity. It concludes our proof of Proposition 2.12 of Chapter 2. **Q.E.D**

II) For curves of degree $i \leq 4$ the Proposition 2.3 of Chapter 2 is immediate. It is sufficient to use the fact that Harnack curves of degree $i \leq 4$ are rigidly isotopy and are L-curves.
(When consider the curve $\mathcal{B}_3$, one can also give the following argument. Assume $\mathcal{B}_4$ is a curve of type $\mathcal{H}^0$ which results from deformation of $\mathcal{B}_3 \cup L$. The curve $\mathcal{B}_3$ has necessarily two real connected components. Noticing that none of the polynomials $x_0^3 + a_1.x_1^3 + a_2.x_2^3$ with $a_1, a_2 \in \mathbf{R}$ is a polynomial of a curve of degree 3 with two real connected components, we get the Proposition 2.3 of Chapter 2 for $\mathcal{B}_3$.)
Q.E.D

Conclusion: Proof of Theorem 2.2 of Chapter 2:

Bringing together Theorem 2.32 of Chapter 2 of the Appendix, Proposition 2.4 of Chapter 2 Proposition 2.12 of Chapter 2, and this last result for curves of degree ≤ 4 we get by induction the Theorem 2.2 of Chapter 2. Q.E.D

The following remark may easily deduced from Theorem 2.2 of Chapter 2.

REMARK 2.16. Let L be a real projective line and S a set of m real points lying on L. Denote by $\mathcal{H}'_m$ a curve of degree m of type $\mathcal{H}^0$ relatively to a real projective line L'. There exists a rigid isotopy of $\mathbf{R}P^2$ which maps $\mathcal{H}'_m$ of type $\mathcal{H}^0$ to a curve $\tilde{\mathcal{H}_m}$ of type $\mathcal{H}^0$ relatively to L such that $\tilde{\mathcal{H}_m} \cap L = S$.

A generalization of Harnack's Method. Harnack curves $\mathcal{H}_i$ of degree $i \leq 4$ are L-curves. Hence, up to rigid isotopy, they result from the successive classical deformation of the union $\mathcal{H}_j \cup L_{j+1}$, $1 \leq j \leq 4$, where L_{j+1} is a real projective line which intersects $\mathcal{H}_j$ in j real points. In Theorem 2.30 of Chapter 2, we extend this property to Harnack curves $\mathcal{H}_m$ of arbitrary degree. Precisely, we prove that up to regular modification of its polynomial any Harnack curve results from the Harnack's method (i.e $L_{j+1} = L$ for any $j \geq 0$). In this way, the Rigid Isotopy Classification Theorem 2.1 of Chapter 2. follows from the Theorem 2.30 of Chapter 2.

Given $\mathcal{H}_m$ a Harnack curve with polynomial $B_m(x_0, x_1, x_2)$, we denote by $\mathcal{B}_i$ the curve with polynomial $B_i(x_0, x_1, x_2) = x_0^i.b_i(x_1/x_0, x_2/x_0)$ where $B_i(x_0, x_1, x_2)$ is the homogeneous polynomial associated to the truncation $b_i(x, y)$ of $b_m(x, y)$ on the monomials $x^\alpha.y^\beta$ with $0 \leq \alpha + \beta \leq i$.
Let us prove the following Proposition:

PROPOSITION 2.17. *Up to rigid isotopy, one can always assume that $\mathcal{B}_i$, $1 \leq i \leq 3$, is of type $\mathcal{H}^0$ relatively to the line at infinity $L := x_0 = 0$.*

proof :
In the proof of the Rigid Isotopy Classification Theorem 2.1 of Chapter 2, we shall use only the fact that up to rigid isotopy, one can always assume that $\mathcal{B}_i$, $1 \leq i \leq 3$, is of type $\mathcal{H}^0$ relatively to the line at infinity $L := x_0 = 0$. Our argumentation is based on the fact that up to rigid isotopy there exists only one M-curve of degree 3.
The proof is based on two lemmas.

I) Noticing that, up to regular modification of $B_m(x_0, x_1, x_2)$, the truncation $B_3(x_0, x_1, x_2)$ is the polynomial of a smooth curve, we shall prove that, up to regular modification of $B_m(x_0, x_1, x_2)$, $B_3(x_0, x_1, x_2)$ is of type $\mathcal{H}^0$. Our argumentation is based on the following statement:
A nonsingular cubic has exactly three real inflection points. These inflection points are collinear. [**6**]

Denote by s_i, $0 \leq i \leq 2$, the real inflection points of B_3 and by L_i, $0 \leq i \leq 2$, the tangent to $\mathcal{B}_3$ at s_i.
Let us prove the following Lemma:

LEMMA 2.18. *For suitable projective coordinates, and up to rigid isotopy of $B_m(x_0, x_1, x_2)$, $\mathcal{B}_3$ results from the classical deformation:*
$B_3(x_0, x_1, x_2) = x_0.B_2(x_0, x_1, x_2) + \epsilon.L_0.L_1.L_2$ where $B_2(x_0, x_1, x_2)$ is of type $\mathcal{H}^0$ relatively to $x_0 = 0$.
($L_i \neq x_0 = 0$, $B_3(0, 1, 1) = \epsilon.L_0.L_1.L_2(0, 1, 1)$)

proof: Consider the Taylor expansion for $B_3(x_0, x_1, x_2)$ at each inflection point s_i, $0 \leq i \leq 2$.

One can assume $s_i \notin \{(x_0 : x_1 : x_2) \in \mathbf{R}P^2 | x_i = 0\}$ and consider $b_{3,i}(x,y) = B_3(x_0,x_1,x_2)|_{x_i=1}$. In a neighborhood $U(s_i)$ of $s_i = (a_i, b_i)$,
$b_{3,i} = \Sigma_{n=0}^{3} \frac{1}{n!} \Sigma_{k=0}^{n} (\frac{\partial^n B_3}{\partial^k x \partial^{n-k} y}(a_i,b_i)(x-a)^k(y-b)^{n-k})$ Each tangent L_i meets $\mathcal{B}_3$ with multiplicity 3. Since $\frac{\partial^2 B_3}{\partial^k x \partial^{2-k} y}(a_i,b_i) = 0$ for any s_i, $0 \leq i \leq 2$, when consider the Taylor expansion of B_3, in any neighborhood $U(s_i) \subset \mathbf{R}P^2$ of the inflection point s_i $B_2(s) = L_i(s)$ for any $s \in U(s_i)$; and thus $B_3(s) - L_i(s) = C_3(s)$ for any $s \in U(s_i)$.
Consider the linear change of coordinates which maps (x_0, x_1, x_2) to $(x'_0 = L_0, x'_1 = L_1, x'_2 = L_2)$. It maps $B_3(x_0,x_1,x_2)$ to $B'_3(x'_0,x'_1,x'_2)$. Let us prove that it also maps C_3 to $\epsilon.x'_0.x'_1.x'_2$. It maps s_i to s'_i and one can assume $s'_i \notin \{(x'_0 : x'_1 : x'_2) \in \mathbf{R}P^2 | x'_i = 0\}$. Consider the Taylor expansion of $b'_{3,i}$ in a neighborhood $U(s'_i) = \{(x'_0 : x'_1 : x'_2) \in \mathbf{C}P^2 | x'_i \neq 0\}$ of s'_i. For any $i \in \{0,1,2\}$, $j,k \neq i$ $j,k \in \{0,1,2\}$, $b'_{3,i}(s) - x'_j(s).x'_k(s) = C'_3(s)$ for any $s \in U(s_i)$. Hence, it follows that $C'_3 = \epsilon.L_0.L_1.L_2$. since $L_0.L_1.L_2(s) - \epsilon.C'_3(s) = 0$ for any $s \in U(s_i)$. It concludes the proof of Lemma 2.18 of Chapter 2. Q.E.D

II)

DEFINITION 2.19. Let $\mathcal{A}$ and $\mathcal{B}$ be smooth algebraic curves of $\mathbf{C}P^2$ with respective order i and j, $i \leq j$. Let $\mathbf{R}\mathcal{A}$,resp. $\mathbf{R}\mathcal{B}$, be the real point set of $\mathcal{A}$, resp. $\mathcal{B}$. We shall say $\mathcal{A}$ is *immersed* in $\mathcal{B}$ if, up to rigid isotopy of $\mathbf{R}\mathcal{B}$, $\mathbf{R}\mathcal{A}$ is embedded in $\mathbf{R}\mathcal{B}$.

DEFINITION 2.20. Given $A(x_0,x_1,x_2) = x_0^i.a_(x_1/x_0, x_2/x_0)$ the polynomial of a curve $\mathcal{A}$ immersed in $\mathcal{B}$. Assume A regular. Denote by $\mathcal{P}_\mathcal{A}$ the pencil of curves over $\mathcal{A}$. (i.e curves with polynomial $x_0^i(a_(x_1/x_0, x_2/x_0) - c)$, $c \in \mathbf{R}$.)

We shall say that $\mathcal{C} \in \mathcal{P}_\mathcal{A}$ is $M - immersed$ (over $\mathcal{A}$) in $\mathcal{B}$ if:

(1) $\mathcal{C}$ is immersed in $\mathcal{B}$
(2) $\mathcal{C}$ has the maximal number of real components a curve of $\mathcal{P}_\mathcal{A}$ immersed in $\mathcal{B}$ may have.

We shall prove in Lemma 2.21 of Chapter 2 that a curve $\mathcal{H}_3$ is immersed in $\mathcal{H}_m$.

LEMMA 2.21. *Let $\mathcal{H}_m$ be the curve with polynomial B_m. Up to regular modification of B_m, the curve $\mathcal{B}_3$ is the curve $\mathcal{H}_3$ where $\mathcal{H}_3$ is immersed in $\mathcal{H}_m$ as the classical deformation of $\mathcal{H}_2 \cup L$ where $L := x_0 = 0$.*

The curve $\mathcal{B}_3$ is determined up to rigid isotopy by its real scheme. The two possible real schemes for $\mathcal{B}_3$ are $\langle J \sqcup 1 \rangle$ and $\langle J \rangle$.
Hence, according to Lemma 2.18 of Chapter 2, up to rigid isotopy of $B_m(x_0,x_1,x_2)$, which is also a rigid isotopy of $B_3(x_0,x_1,x_2)$, one can assume that $B_3(x_0,x_1,x_2)$ results from the classical deformation of the union of a line L with a curve $\mathcal{B}_2$ of degree 2 with real scheme $\langle 1 \rangle$; the deformation is directed to the union of the tangents at real inflection points of $\mathcal{B}_3$. Real inflection points of $\mathcal{B}_3$ are points of L. It is not hard to see that if none of these points belongs to the inner of $\mathcal{B}_2$, then $\mathcal{B}_3$ is of type $\mathcal{H}^0$ relatively L. Otherwise, if at least one of this point belongs to the inner of $\mathcal{B}_2$,then $\mathcal{B}_3$ has real scheme $\langle J \rangle$.

Denote L the line infinity $x_0 = 0$, and let $b_3(x, y) = B_3(1, x_1, x_2)$ be the affine polynomial associated to $B_3(x_0, x_1, x_2)$. Consider the pencil $x_0^m.(b_3(x_1/x_0, x_2/x_0) - c)$. For any $i \geq 4$, let $C_i(x_1, x_2)$ be the polynomial of degree i in the variables x_1, x_2 such that: $B_m(x_0, x_1, x_2) = x_0^{m-3}.B_3(x_0, x_1, x_2) + \Sigma_{i=4}^m x_0^{m-i}.C_i(x_1, x_2)$.
Consider the pencil $x_0^{m-3}.(b_3(x_1/x_0, x_2/x_0) - c)$. Assume B_3 regular. For any critical point $(1, x_{0,1}, x_{0,2})$ of $B_3(1, x_1, x_2) = b_3(x_1, x_2)$, choose its representative in S^2 $\frac{1}{(1+x_{0,1}^2+x_{0,2})^{1/2}}(1, x_{0,1}, x_{0,2})$ In such a way, it is easy to see that a curve of degree 3 is immersed in $\mathcal{B}_m$. Since $\mathcal{B}_m$ has the maximal number of real components a curve of degree m may have, an M-curve of degree 3 is M-immersed in $\mathcal{B}_m$. Indeed, if the curve $\mathcal{B}_3$ is not an M-curve, it is an easy consequence of the Petrovskii's theory that as c varies from $[-\Sigma_{j=4}^m ||C_j||, \Sigma_{j=4}^m ||C_j||]$ the real point set of the curve $\mathcal{B}_3$ undergoes at least one Morse modification (i.e at least one critical point passes through its critical value.) According to Lemma 2.18 of Chapter 2, this is equivalent to say that any union of lines $L_i \cup L$ $1 \leq i \leq 3$ is perturbed in such a way none of the resulting real branches intersects the oval $\mathcal{B}_2$. It follows that, up to regular deformation of $B_m(x_0, x_1, x_2)$, $B_3(x_0, x_1, x_2)$ is of type $\mathcal{H}^0$. Since curves of degree $m \leq 3$ are defined up to rigid isotopy by their real scheme, up to rigid isotopy of $B_m(x_0, x_1, x_2)$, $B_2(x_0, x_1, x_2)$, and in this way also $B_1(x_0, x_1, x_2)$ are of type $\mathcal{H}^0$. It concludes the proof of Lemma 2.21 of Chapter 2. Q.E.D

The Proposition 2.17 of Chapter 2 is a straightforward consequence of the Lemma 2.21 of Chapter 2. Q.E.D

Denote Ω_j the set of M-curves with real scheme:
-for even $j = 2k$

$$\langle 1\langle\alpha\rangle \sqcup \beta\rangle$$

with $\alpha + \beta = \frac{(j-1)(j-2)}{2}$
-for odd $j = 2k + 1$

$$\langle J \sqcup \gamma\rangle$$

with $\gamma = \frac{(j-1)(j-2)}{2}$

PROPOSITION 2.22. *Let $\mathcal{H}_m$ be a Harnack curve of degree $m \geq 4$. Denote by B_m its polynomial. Up to regular modification of B_m, the curve $\mathcal{H}_m$ results from the deformation of the union $\mathcal{A}_{m-i} \cup \mathcal{H}_i$, $i \leq 3$, where $\mathcal{A}_{m-i} \in \Omega_{m-i}$ and $\mathcal{H}_i$ is a Harnack curve of degree $i \leq 3$.*

proof:

Set $L := x_0 = 0$. It follows from Proposition 2.17 of Chapter 2 that one can set

$$B_m(x_0, x_1, x_2) = A_{m-i}(x_0, x_1, x_2).B_i(x_0, x_1, x_2) + C_m(x_0, x_1, x_2)$$

where $B_i(x_0, x_1, x_2)$ is a Harnack of degree $i \leq 3$ and type $\mathcal{H}^0$ relatively to L and $A_{m-i}(x_0, x_1, x_2) = x_0^{m-i}$.
Let us prove that there exists a regular modification of $B_{m,t}(x_0, x_1, x_2)$, $t \in [0, 1]$,

$$B_{m,0}(x_0, x_1, x_2) = B_m(x_0, x_1, x_2)$$

$$B_{m,t}(x_0,x_1,x_2) = A_{m-i,t}(x_0,x_1,x_2).B_{i,t}(x_0,x_1,x_2) + C_{m,t}(x_0,x_1,x_2)$$

such that:

(1) $B_{i,t}(x_0,x_1,x_2)$ is a regular modification of $B_i(x_0,x_1,x_2)$ the Harnack polynomial of a Harnack curve $\mathcal{H}_i$,
(2) $A_{m-i,1}$ is the polynomial of a smooth curve of degree $m-i$ ($A_{m-i,0} = x_0^{m-i}$)
(3) The polynomial $B_{m,1}$ is the polynomial of a curve $\mathcal{H}_m$ which results from the classical deformation of the union of $\mathcal{A}_{m-i,1} \cup \mathcal{H}_i$ with polynomial $A_{m-i,1} \cup B_{i,1}$

(It is obvious that the regular modification $B_{m,t}$ of B_m is not regular on A_{m-i}) Our proof is based on the fact that curves of degree $i \leq 4$ are determined up to rigid isotopy by their real schemes and may be realized as L-curves (i.e may realized by classical small perturbation of i lines in general position) [**8**]. We shall consider immersion (see the definition 2.19 of Chapter 2 of the proof of Proposition 2.17 of Chapter 2), of such curves in the curve $\mathcal{H}_m$ with polynomial B_m.

We shall consider curves $\mathcal{H}_m$ of degree $m \leq 7$ and degree $m \geq 8$ separately.

1)
Let us consider curves $\mathcal{H}_m$ of degree $m \leq 7$. According to the Proposition 2.17 of Chapter 2, up to rigid isotopy, the polynomial B_m of $\mathcal{H}_m$ is of the form

$$B_m(x_0,x_1,x_2) = x_0^{m-3}.B_3(x_0,x_1,x_2) + C_m(x_0,x_1,x_2)$$

with $m \leq 7$ where B_3 is the polynomial of a Harnack curve of degree 3.
- One can deform the polynomial $A_{m-3,0} = x_0^{m-3}$ into the polynomial $\Pi_{j=1}^{m-3}(x_{0,j})$ of $m-3$ lines in general position in such a way that the path $B_{m,t}$, $t \in [0,t_0]$, is rigid isotopy from $B_{m,0} = B_m$ to B_{m,t_0}.

$$B_{m,t_0}(x_0,x_1,x_2) = \Pi_{j=1}^{m-3}(x_{0,j}).B_3(x_0,x_1,x_2) + C_m(x_0,x_1,x_2)$$

- According to the Lemma 2.21 of Chapter 2, one can assume that the curve $\mathcal{H}_3$ with polynomial B_3 is immersed in $\mathcal{H}_m$ as the classical deformation of $\mathcal{H}_2 \cup L$ where $L := x_0 = 0$.
Therefore, one can consider a rigid isotopy $B_{m,t}$, $t \in [t_0,t_1]$,

$$B_{m,t_1} = B_3.B_{m-3} + C_m$$

where B_{m-3} is a regular polynomial of an L-curve which results from the deformation of $\Pi_{j=1}^{m-3}(x_{0,j})$.

Let $||B_3||$ be the norm of B_3 and let $b_{m-3}.||B_3||$ be the affine polynomial associated to $B_{m-3}.||B_3||$. Consider the pencil of curves $x_0^m(b_{m-3}.||B_3|| - c)$.
For any critical point (x_0,x_1,x_2) of $B_{m-3}.||B_3||$ choose its representative in S^2 $\frac{1}{(1+x_{0,1}^2+x_{0,2}^2)^{1/2}}(1,x_{0,1},x_{0,2})$. As c varies from $[-||C_m||, ||C_m||]$, for any critical point of b_{m-3} which goes through its critical value; the real point set of a curve of the pencil undergoes a Morse modification. Since $\mathcal{H}_m$ has the maximal number of real connected components a curve of degree m may have, it is not hard to see that an M-curve of degree $m-3$ is M-immersed in $\mathcal{H}_m$. Up to rigid isotopy of B_m, it is realized as the L-curve which results from the perturbation of the union of $m-3$

lines with polynomial $\Pi_{j=1}^{m-3}(x_{r,j})$.
- It follows that there exists a rigid isotopy $B_{m,t}$, $t \in [0,1]$ from $B_{m,0}$ to

$$B_{m,1} = B_3.B_{m-3} + C_{m,1}$$

where B_{m-3} is the polynomial of M-curve of degree $m-3 \leq 7$; and the curve the curve $\mathcal{H}_m$ results from classical small deformation of $\mathcal{H}_{m-3} \cup \mathcal{H}_3$ with polynomial $B_{m-3}.B_3$. Since any curve $\mathcal{H}_{m-3}$ of degree $m-3 \leq 4$ belongs to Ω_{m-3}, it proves the Proposition 2.22 of Chapter 2 for curves $\mathcal{H}_m$ of degree $m \leq 7$.

2)
Let us consider curves $\mathcal{H}_m$ of degree $m \geq 8$. According to the Proposition 2.17 of Chapter 2, up to rigid isotopy, the polynomial B_m of $\mathcal{H}_m$ is of the form

$$B_m(x_0,x_1,x_2) = x_0^{m-3}.B_3(x_0,x_1,x_2) + C_m(x_0,x_1,x_2)$$

where B_3 is the polynomial of a Harnack of degree 3 and $m-3 = 4.r+s$ with $r, s \in \mathbf{N}, s \leq 3$.

Set $A_{m-3,0} = x_0^{m-3} = x_0^{4r+s} = (x_0^4)^r.x_0^s = (x_0^4 + ...x_0^4).x_0^s$. For any of the n^{th} $(1 \leq n \leq r)$ product x_0^4 of the polynomial $(x_0^4)^r = x_0^4 + ...x_0^4)$ consider the deformation of x_0^4 into the polynomial $\Pi_{j=1}^4(x_{n,j})$ of the union of four lines in generic position. In the same way, consider the deformation of x_0^s into the polynomial $\Pi_{j=1}^s(x, r+1, j)$ of s lines in generic position.
- For any of these union of 4 (resp, $s \leq 4$), lines in generic position, consider the rigid isotopy $B_{m,t}$, $t \in [0, t_0]$ from $B_{m,0} = B_m$ to B_{m,t_0}.
For $1 \leq n \leq r$,

$$B_{m,t_0}(x_0,x_1,x_2) = x_0^{4(r-1)+s}.\Pi_{j=1}^4(x_{n,j}).B_3(x_0,x_1,x_2) + C_m(x_0,x_1,x_2)$$

(resp,

$$B_{m,t_0}(x_0,x_1,x_2) = x_0^{4r}.\Pi_{j=1}^s(x_{r+1,j}).B_3(x_0,x_1,x_2) + C_m(x_0,x_1,x_2))$$

From an argumentation analogous to the one given in **1**), it follows that, up to regular deformation of B_m, any of the union of 4 (resp, s) lines in generic position may be deformed into an L-curve of degree 4 (resp, s) which is an M-curve immersed in $\mathcal{H}_m$. Denote by $\mathcal{H}_{4,n}$, $1 \leq n \leq r$, (resp, $\mathcal{H}_s$) the corresponding curves. Denote by $B_{4,n}$ (resp, B_s) the polynomial of $\mathcal{H}_{4,n}$ (resp, $\mathcal{H}_s$). **-** It follows that there exists a rigid isotopy $B_{m,t}$, $t \in [0, t_1]$, from $B_{m,0} = B_m$ to B_{m,t_1}.

$$B_{m,t_1}(x_0,x_1,x_2) = \Pi_{n=1}^r(B_{4,n}).B_s.B_3 + C_{m,t_1}(x_0,x_1,x_2)$$

DEFINITION 2.23. Call *succesive* small perturbation of a finite union of $\cup_{i=1}^n \mathcal{A}_i$ of curve $\mathcal{A}_i$ the result of the following recursive classical small perturbation of the union of two curves:

(1) the union $\mathcal{A}_1 \cup \mathcal{A}_2$ is a singular curve all of whose singular points are crossings; the classical small perturbation $\mathcal{A}_1 \cup \mathcal{A}_2$ leads to a curve $\mathcal{B}_2$
(2) For $2 \leq i \leq n-1$, the classical small perturbation $\mathcal{B}_i \cup \mathcal{A}_{i+1}$ leads $\mathcal{B}_{i+1}$ such that the union $\mathcal{B}_{i+1} \cup \mathcal{A}_{i+2}$ is a singular curve all of whose singular points are crossings.

Let us prove that a curve $\mathcal{A}_{m-3} \in \Omega_{m-3}$ of degree $m-3$ which results from a successive classical small deformation of the union $\cup_{n=1}^r \mathcal{H}_{4,r} \cup \mathcal{H}_s$ is M-immersed

in $\mathcal{H}_m$.

Let us consider the union of curves $\cup_{n=1}^{r}\mathcal{H}_{4,n}\cup\mathcal{H}_s$ with polynomial $\Pi_{n=1}^{r}B_{4,n}.B_s$. Up to rigid isotopy of B_m, one can assume $\mathcal{H}_{4,n}=\mathcal{H}_{4,1}$, $2\leq n\leq r$, where $\mathcal{H}_{4,1}$ is the L-curve of degree 4 which results from the deformation of the union of four lines in generic position with polynomial $\Pi_{j=1}^{4}x_{1,j}$. One can also assume that $\mathcal{H}_s$, $s\leq 4$,is the L-curve of degree s which results from the classical deformation of the union of s lines with polynomial $\Pi_{j=1}^{s}x_{1,j}$.
Since $\mathcal{H}_m$ is an M-curve, for any i , $1\leq i\leq r$, there exists an M-curve of degree $4i+s$ immersed in $\mathcal{H}_m$ as the result of the successive classical perturbation of $\cup_{k=1}^{i}\mathcal{H}_{4,k}\cup\mathcal{H}_s$. Denote by $\mathcal{B}_{s+4i}$ the M-curve of degree $4+si$ immersed in $\mathcal{H}_m$ as the classical perturbation of $\cup_{k=1}^{i}\mathcal{H}_{4,k}\cup\mathcal{H}_s$.

The curve $\mathcal{B}_{4+s}$ immersed in $\mathcal{H}_m$ belongs to Ω_{4+s}. Indeed, none of the pair of ovals $(\mathcal{O},\mathcal{O}')\in(\mathbf{R}\mathcal{H}_{4,1},\mathbf{R}\mathcal{H}_s)$ is injective. In the same way, since $\mathcal{H}_{4,n}=\mathcal{H}_{4,1}$ for $2\leq n\leq r$ none of the pair of ovals $(\mathcal{O},\mathcal{O}')\in(\mathbf{R}\mathcal{B}_{s+4i},\mathbf{R}\mathcal{H}_{4,i+1})$ is an injective pair of ovals. Hence, at any step of the successive classical small deformation we get an M-curve $\mathcal{B}_{s+4i}\in\Omega_{s+4i}$.

For clarity, let us verify that, up to rigid isotopy of B_m, the successive classical deformation of $\cup_{k=1}^{r}\mathcal{H}_{4,k}\cup\mathcal{H}_s$ as curve immersed in $\mathcal{H}_m$ does not depend on the order of the deformations.

Let us notice that:

Remark 2.24. The classical small deformation of the union of two M-curves $\mathcal{A}\cup\mathcal{B}$ leads to an M curve only if common points of $\mathcal{A}$ and $\mathcal{B}$ belong to one connected real connected of $\mathcal{A}$ and one connected real connected of $\mathcal{B}$.

proof: Otherwise, it would lead to contradiction with the number of real connected components of $\mathcal{H}_m$. Q.E.D

According to remark 2.24 of Chapter 2, there exists $\mathcal{O}\in\mathbf{R}\mathcal{H}_{4,1}$ such that any two curves immersed in $\mathcal{H}_m$ as the successive classical deformation $\cup_{n=1}^{l}\mathcal{H}_{4,n}$ and respectively $\cup_{n=1}^{l'}\mathcal{H}_{4,n}$, $l \not l', 2\leq l,l'\leq r$ (where $\mathcal{H}_{4,2}\neq\mathcal{H}_{4,1}$) intersects $\mathcal{H}_{4,1}$ in $\mathcal{O}$. In the same way, there exists $\mathcal{O}\in\mathbf{R}\mathcal{H}_s$ (if s is odd, $\mathcal{O}=J$ the odd component of $\mathcal{H}_s$) such that any two curves immersed in $\mathcal{H}_m$ as the successive classical deformation $\cup_{n=1}^{l}\mathcal{H}_{4,n}$ and $\cup_{n=1}^{l'}\mathcal{H}_{4,n}$, $l\neq l'$, $2\leq l,l'\leq r$ (where $\mathcal{H}_{4,2}\neq\mathcal{H}_{4,1}$) intersect $\mathcal{H}_s$ in $\mathcal{O}$.

Hence, the M-curve $\mathcal{A}_{m-3}$ immersed in $\mathcal{H}_m$ as the result of the successive classical small deformation of $\cup_{k=1}^{r}\mathcal{H}_{4,k}\cup\mathcal{H}_s$ is well defined. Besides, it belongs to Ω_{m-3}.

- It follows that there exists a rigid isotopy $B_{m,t}$, $t\in[0,1]$ from $B_{m,0}$ to

$$B_{m,1}=B_3.B_{m-3}+C_{m,1}$$

where B_{m-3} is the polynomial of $\mathcal{A}_{m-3}\in\Omega_{m-3}$ and $\mathcal{H}_m$ results from classical small deformation of the curve $\mathcal{A}_{m-3}\cup\mathcal{H}_3$ with polynomial $B_{m-3}.B_3$.
It concludes the proof of Proposition 2.22 of Chapter 2. Q.E.D

In the subsection 2 of Chapter 2, we have proven that if $\mathcal{H}_n$ is of type $\mathcal{H}^0$, then, up to regular modification of its polynomial B_n, the polynomial B_{n-1} is also

of type $\mathcal{H}^0$. Let us now in Proposition 2.25 of Chapter 2 prove the converse.

PROPOSITION 2.25. *Let $\mathcal{H}_n$ be a Harnack curve of degree n, $n \geq 4$, and $B_n(x_0,x_1,x_2) = x_0^n.b_n(x_1/x_0,x_2/x_0)$ its polynomial. Denote by $B_{n-1}(x_0,x_1,x_2) = x_0^{n-1}.b_{n-1}(x_1/x_0,x_2/x_0)$ the homogeneous polynomial associated to the truncation $b_{n-1}(x,y)$ of $b_n(x,y)$ on the monomials $x^\alpha.y^\beta$ with $0 \leq \alpha+\beta \leq n-1$.*
If $B_{n-1}(x_0,x_1,x_2)$ is of type $\mathcal{H}^0$ relatively to the line $L_n := x_0 = 0$, then up to regular modification of B_n, there exists a line L_{n+1} with the property that $\mathcal{H}_n$ is of type $\mathcal{H}^0$ relatively to L_{n+1}.

proof :
Our proof uses an argumentation similar to the one given in the subsection 2 of Chapter 2 to prove that if $\mathcal{H}_n$ is of type $\mathcal{H}^0$ then up to regular modification of its polynomial B_n, the curve $\mathcal{B}_{n-1}$ is also of type $\mathcal{H}^0$.
Let $L_n := x_0 = 0$ and $\mathcal{M}$ be a neighborhood of the line L_n. The curve $\mathcal{H}_n$ with polynomial B_n results from deformation of $\mathcal{H}_{n-1} \cup L_n$

$$B_n = B_{n-1}(x_0,x_1,x_2).x_0 + C_n(x_0,x_1,x_2)$$

where $C_n(x_0,x_1,x_2)$ is homogeneous of degree n. Let $U(s) \subset \mathbf{R}P^2$ be a neighborhood of a singular point of $\mathbf{R}\mathcal{H}_{n-1} \cup \mathbf{R}L_n$ such that $\mathbf{R}\mathcal{H}_n \cap U(s) \neq \emptyset$. According to the topology of $\mathcal{H}_n$, for each singular point s there exists a homeomorphism $h : U(s) \to D^1 \times D^1$ such that $h(\mathbf{R}\mathcal{H}_{n-1} \cup L_n) = D^1 \times 0 \cup 0 \times D^1$ and $h(\mathbf{R}\mathcal{H}_n \cap U(s)) = \{(x,y) \in D^1 \times D^1 | xy = 1/2\}$. Since the polynomial of $\mathcal{H}_n$ is smooth, it follows by continuity, that real connected components of $\mathbf{R}\mathcal{H}_n$ define entirely and uniquely the homomorphism h.
We shall enlarge the previous description of $\{x \in \mathbf{R}P^2 | B_n(x) = 0\}$ in $U(s)$ to a description of in U_s where

$$U_s = \{z =< u,p >= (u_0.p_0 : u_1.p_1 : u_2.p_2) \in \mathbf{C}P^2 | u = (u_0 : u_1 : u_2) \in U^3_{\mathbf{C}}, p = (p_0 : p_1 : p_2) \in U(s)\}$$

To this end, let us prove the following Lemma 2.26 of Chapter 2.

LEMMA 2.26. *Let $B_n(x_0,x_1,x_2) = x_0^n.b_n(x_1/x_0,x_2/x_0)$ be a Harnack polynomial of degree n. Denote by $B_{n-1}(x_0,x_1,x_2) = x_0^{n-1}.b_{n-1}(x_1/x_0,x_2/x_0)$ the homogeneous polynomial associated to the truncation $b_{n-1}(x,y)$ of $b_n(x,y)$ on the monomials $x^\alpha.y^\beta$ with $0 \leq \alpha+\beta \leq n-1$.*
Assume that B_{n-1} is of type $\mathcal{H}^0$ relatively to $L := x_0 = 0$.

1. *there exists $\mathcal{M}$ neighborhood of L such that the polynomial $B_n^\Delta(x_0,x_1,x_2) = (x_0^n.b_n^\Delta(x_1/x_0,x_2/x_0)$ where b_n^Δ is the truncation of b_n on monomials of homogeneous degree $n-2 \leq i \leq n$ is ϵ-sufficient for $B_n(x_0,x_1,x_2)$.*
2. *Up to regular modification of B_n, there exist $\mathcal{M}_\epsilon \subset \mathcal{M}$, $\mathcal{M}_\epsilon = \cup_{s \in \mathbf{R}\mathcal{H}_{m-1} \cap L} U_s$ ϵ-tubular neighborhood of L and a polynomial $\tilde{B}_n$ with the following properties:*
   -$\tilde{B}_n$ is ϵ-sufficient for B_n in $\mathcal{M}_\epsilon$
   -for any U_s, $U_s \subset \mathcal{M}_\epsilon$, the truncation $\tilde{B}_n^S$ of $\tilde{B}_n^S$ (on four monomials $x^cy^dz^2, x^{c+1}y^dz, x^cy^{d+1}z, x^{c+1}y^{d+1}$ with $c+d+2 = n$) is ϵ-sufficient for $\tilde{B}_n$ (and thus for B_n) in U_s. (For any $s \neq s'$, such that the truncation of $\tilde{B}^n$ on $x^cy^dz^2, x^{c+1}y^dz, x^cy^{d+1}z, x^{c+1}y^{d+1}$ (resp,on $x^{c'}y^{d'}z^2, x^{c'+1}y^{d'}z, x^{c'}y^{d'+1}z, x^{c'+1}y^{d'+1}$) is ϵ-sufficient for $\tilde{B}_n$ (and thus for B_n) in U_s (resp, in $U(s')$, $(c,d) \neq$

(c', d'))

REMARK 2.27. Note that if the curve $\mathcal{H}_n$ is not of type $\mathcal{H}^0$ (relatively to L), then $\mathcal{M}_\epsilon \subset \mathcal{M}$, $\mathcal{M}_\epsilon \neq \mathcal{M}$.

proof: Assume B_{n-1} of type $\mathcal{H}^0$ relatively to $L_n := x_0 = 0$. According to the proof of Morse Lemma, local coordinates defined in a neighborhood $U(s)$ of a point s depend principally on the first derivative and the second derivative of the function b_n around this point. Hence, the truncation of $b_n^\Delta(x, y)$ on monomials of homogeneous degree $n-2 \leq i \leq n$ is ϵ-sufficient for $b_n(x, y)$.
According to Morse Lemma [**15**], in neighborhood $U(s)$ of any non-degenerate singular point s of $x_0.B_{n-1}$ one can choose local coordinates system z_1, z_2 with $z_1(s) = 0, z_2(s) = 0$ and $x_0.B_{n-1} = z_1.z_2$. Real connected components of $\mathbf{R}\mathcal{H}_n$ define entirely and uniquely the homomorphism h defined in $U(s)$ $h : U(s) \to D^1 \times D^1$ such that $h(\mathbf{R}\mathcal{H}_{n-1} \cup L_n) = D^1 \times 0 \cup 0 \times D^1$ and $h(\mathbf{R}\mathcal{H}_n \cap U(s)) = \{(x, y) \in D^1 \times D^1 | xy = 1/2\}$. In such a way, in a neighborhood $U(s)$ of a point s, one can choose local coordinates system y_1, y_2 with $y_1(s) = 0, y_2(s) = 0$; $B_n = y_1.y_2 + B_n(s)$. In $U(s)$, any point s is a local extremum of b_n. Any singular point s of $\mathbf{R}B_{n-1} \cup \mathbf{R}L_n$ belongs to the line $x_0 = 0$, hence it is a local extremum of the function $b_n(y) = b_n(0, x_1/x_2, 1)$. One can assume without loss of generality, that any point s does not belong to the line $x_2 = 0$. On such assumption gradient trajectories of $b_n(x_0, x_1, 1)$ in $U(s)$ give the direction of the perturbation of the crossing s. Thus, up to regular modification of B_n and for sufficiently small $\epsilon > 0$, there exist $U(s) \subset \mathcal{M}_\epsilon$ and a polynomial $\tilde{b}_n^S$ on four monomials ($x^c y^d, x^{c+1} y^d, x^c y^{d+1}, x^{c+1} y^{d+1}$ with $c+d+2 = n$) which is ϵ-sufficient for $B_n(1, x_1, x_2)$ in $U(s) \subset \mathcal{M}_\epsilon$; $\tilde{b}_n^S(x, y) = l(x, y) + k(x, y)$ with $l(x, y) = a_{c,d} x^c y^d + a_{c+1,d} x^{c+1} y^d$
$k(x, y) = a_{c,d+1} x^c y^{d+1} + a_{c+1,d+1} x^{c+1} y^{d+1}$ with $c + d + 2 = n$. From this last observation, we deduce the definition of $\tilde{b}_n$. Q.E.D

Let us consider the polynomial $\tilde{b}_n$ with truncation $\tilde{b}_n^S$ ϵ-sufficient for $B_n(1, x_1, x_2)$ in U_s.
$\tilde{b}_n^S(x, y) = l(x, y) + k(x, y)$ with $l(x, y) = a_{c,d} x^c y^d + a_{c+1,d} x^{c+1} y^d$
$k(x, y) = a_{c,d+1} x^c y^{d+1} + a_{c+1,d+1} x^{c+1} y^{d+1}$ with $c + d + 2 = n$.

Up to modify the coefficients $a_{c,d}, a_{c,d+1}, a_{c+1,d}, a_{c+1,d+1}$ if necessary, the point $s = (x_0, y_0)$ is, up to homeomorphism, a critical point of the function $\frac{l(x,y)}{k(x,y)}$. Hence, it follows from the equalities $l(x, -y) = -l(x, y)$, $k(x, -y) = k(x, y)$,
$\frac{\partial l}{\partial x}(x, -y) = -\frac{\partial l}{\partial x}(x, y)$, $\frac{\partial l}{\partial y}(x, -y) = \frac{\partial l}{\partial y}(x, y)$,
$\frac{\partial k}{\partial x}(x, -y) = \frac{\partial k}{\partial x}(x, y)$, $\frac{\partial k}{\partial y}(x, -y) = -\frac{\partial k}{\partial y}(x, y)$
that $(x_0, -y_0)$ is also a critical point of the function $\frac{l(x,y)}{k(x,y)}$. In such a way, we define a map which maps the set $\mathcal{S}$ of singular points of $\mathcal{H}_{n-1} \cup L_n$ to a set $\mathcal{P}$ of $n-1$ real points. In a neighborhood $U(p)$ of $p \in \mathcal{P}$, $\mathbf{R}\mathcal{H}_m$ is desingularized crossing i.e consists of two real arcs γ, γ'. The set $\mathcal{S}$ is a set of aligned points. Hence, up to regular modification, the set $\mathcal{P}$ is also a set of aligned points which belong to a line L. Let $\mathcal{M}_\epsilon = \mathcal{M}^+ \cup \mathcal{M}^-$,$\partial\mathcal{M}^\pm = L_1^\pm \cup L_2^\pm$, be a neighborhood of the line L; $\cup_{p \in \mathcal{P}} U(p) \subset \mathcal{M}$.
In the case $n = 2k$, any pair of arcs of $\mathbf{R}\mathcal{H}_{2k} \cap U(p)$, $p \in \mathcal{P}$, is such that one arc belongs to the non-empty oval of $\mathbf{R}\mathcal{H}_{2k}$. The non-empty oval is connected. Hence,

it intersects a projective line L_1 in $2k$ points.
In the case $n = 2k+1$, any pair of arcs of $\mathbf{R}\mathcal{H}_{2k+1} \cap U(p)$, $p \in \mathcal{P}$, is such that at least one arc belongs to the odd component of $\mathbf{R}\mathcal{H}_{2k+1}$. The odd component is connected. Hence, it intersects a projective line L_1 in $2k+1$ points.
In such a way, we get that if $\mathcal{H}_{n-1}$ is of type $\mathcal{H}^0$ relatively to L_n, there exists $L_1 := L_{n+1}$ such that the curve $\mathcal{H}_n$ is of type $\mathcal{H}^0$ relatively to $L_1 = L_{n+1}$. This concludes our proof of Proposition 2.25 of Chapter 2. Q.E.D

Let us prove in Theorem 2.28 of Chapter 2 , that up to regular modification of the Harnack polynomial $B_m(x_0, x_1, x_2)$, there exists a sequence of lines L_{i+1}, $1 \leq i \leq m$, such that B_i is of type $\mathcal{H}^0$ relatively to the line L_{i+1} and describe relative position of these lines. We shall call rotation of $\mathbf{R}P^2$ the transformation of $\mathbf{R}P^2$ which extends continuously a usual rotation of $\mathbf{R}^2$. Relative position of lines L_i, $1 \leq i \leq m+1$ is described by means of rotations of $\mathbf{R}P^2$.

THEOREM 2.28. *Let $\mathcal{H}_m$ be a Harnack curve of degree m. Up to rigid isotopy, there exists a sequence of lines $\mathcal{H}_1 = L_1$, L_2, ..., L_{m+1} such that $\mathcal{H}_i$ is of type $\mathcal{H}^0$ relatively to L_{i+1} and $\mathcal{H}_{i+1}$ results from the deformation of $\mathcal{H}_i \cup L_{i+1}$.*
The sequence of lines L_i may be ordered in three different ways:

(1) *There exist $p \in L_2 \cap L_j$, $3 \leq j \leq m+1$, and a rotation of center p and angle θ (where θ may be chosen arbitrarily small) which maps $\mathbf{R}L_i$ to $\mathbf{R}L_{i+1}$.*
(2) *There exists $p \in L_2 \cap L_j$, $3 \leq j \leq m+1$, such that:*
for i, $2 \leq i \leq 2.[(m+1)/2] - 2$ the rotation of center p and angle θ maps $\mathbf{R}L_i$ to $\mathbf{R}L_{i+2}$. for i, $1 \leq i \leq 2.[(m+1)/2] - 2$ the rotation of center p and angle $-\theta$ maps $\mathbf{R}L_{i+1}$ to $\mathbf{R}L_{i+3}$ (θ may be chosen arbitrarily small)
- [m/2] denotes the integer n, $(m-1)/2 \leq n \leq m/2$ -
(3) *There exists $p \in L_2 \cap L_j$, $3 \leq j \leq m+1$, such that:*
for i, $2 \leq i \leq l$, arrangement of lines L_{i+1} is given by 2
for i, $l \leq i \leq m+1$ arrangement of lines L_{i+1} is given by 1

proof: In a first part, we shall prove Theorem 2.28 of Chapter 2 for curves $\mathcal{H}_m$ of even degree m. Then, we shall extend our argumentation to Harnack curves of odd degree.

We shall use the rigid isotopy classification of Harnack curves of type $\mathcal{H}^0$ given in Theorem 2.2 of Chapter 2. The following remark may be easily deduced from the Harnack's construction of curves. (see [**11**], [**18**], tables of isotopy types of Harnack M-curves)

REMARK 2.29. The Harnack curve $\mathcal{H}_{n+1}$ is the only M-curve of degree $n+1$ with:
-for even $n+1$ only one non-empty oval
- for odd $n+1$ only empty ovals
which results from the classical deformation of the union $\mathcal{H}_n \cup L_{n+1}$ of a Harnack $\mathcal{H}_n$ of type $\mathcal{H}^0$ relatively to a line L_{n+1}.

I) Harnack Curves of even degree $\mathcal{H}_{2k}$
Our proof is based on the construction of a sequence of curves $\mathcal{A}_{2i} \in \Omega_{2i}$, $1 \leq i \leq$

$k-1$ such that the deformation of $\mathcal{A}_{2i} \cup \mathcal{H}_{2k-2i}$ leads to the curve $\mathcal{H}_{2k}$.
According to Proposition 2.22 of Chapter 2, one can assume that $\mathcal{H}_{2k}$ results from the deformation of $\mathcal{A}_{2k-2} \cup \mathcal{H}_2$ where $\mathcal{A}_{2k-2} \in \Omega_{2k-2}$. It follows from the Bezout's theorem Morse Lemma and the real components of $\mathbf{R}\mathcal{H}_{2k}$ that the $(2k-2).2$ common points of $\mathcal{A}_{2k-2}$ and $\mathcal{H}_2$ belong to their non-empty ovals. We shall consider connected arcs I_j, $3 \leq j \leq 2k$ of the non-empty oval of $\mathcal{A}_{2k-2}$; $\cup I_j \supset \mathbf{R}\mathcal{H}_2 \cap \mathcal{A}_{2k-2}$ and identify these arcs I_j with lines $\mathbf{R}L_j$. In this way, we shall construct a sequence of curves $\mathcal{H}_j$ of type $\mathcal{H}^0$ relatively to L_{j+1} with the property that curves $\mathcal{H}_{2k-2i}$ are such that $\mathcal{A}_{2i} \cup \mathcal{H}_{2k-2i}$ leads to the curve $\mathcal{H}_{2k}$.
We shall call *extreme point* a point of the intersection $\mathbf{R}\mathcal{A}_{2k-2} \cap \mathbf{R}\mathcal{H}_2$ the two points connected by an arc of $\mathbf{R}\mathcal{A}_{2k-2}$ which does not contain other points $\mathbf{R}\mathcal{A}_{2k-2} \cap \mathbf{R}\mathcal{H}_2$.
We shall call *extreme arc* an arc which contains an extreme point.

Let us detail the construction of the sequences of curves $\mathcal{A}_{2i} \in \Omega_{2i}$, $1 \leq i \leq k-1$. Any Harnack curves $\mathcal{H}_2$ are rigidly isotopic. Therefore, without changing relative position of real connected components of $\mathcal{A}_{2k-2}$ and $\mathcal{H}_2$, we may assume $\mathcal{H}_2$ of type $\mathcal{H}^0$ relatively to an (extreme) arc $I_3 \approx \mathbf{R}L_3$ of $\mathcal{A}_{2k-2}$. (Choose I_3 of small lenght. up to glue its extremities, we may identify I_3 with a projective line $\mathbf{R}L_3$.) In such a way, according to the remark 2.29 of Chapter 2, the deformation of $I_3 \cup \mathbf{R}\mathcal{H}_2$ leads necessarily to the Harnack curve $\mathcal{H}_3$. (Otherwise, it would lead to contradiction with the real connected components of $\mathcal{H}_{2k}$.) Moreover, it follows from Proposition 2.25 of Chapter 2, that the curve $\mathcal{H}_3$ is of type $\mathcal{H}^0$ relatively to one line. According to Theorem 2.2 of Chapter 2 and remark 2.16 of Chapter 2, we may, up to regular modification, identify (without changing relative position of real connected components of $\mathbf{R}\mathcal{A}_{2k-2}$ and $\mathbf{R}\mathcal{H}_2$) this line with an other arc I_4 of the non-empty oval of the curve $\mathcal{A}_{2k-2}$. The arc I_4 contains, besides a set of 2 consecutive points which are up to rigid isotopy points of $\mathbf{R}\mathcal{A}_{2k-2} \cap \mathbf{R}\mathcal{H}_2$, an other point. Identifying arcs with lines $I_3 \approx \mathbf{R}L_3$, $I_4 \approx \mathbf{R}L_4$, this last point is provided by the intersection $L_3 \cap L_4$. From an argument analogous to the previous one, we get a curve $\mathcal{H}_4$ of type $\mathcal{H}^0$ relatively to an arc $I_5 \approx \mathbf{R}L_5$. By induction, it follows that $\mathcal{H}_{2k}$ results from the deformation of $\mathcal{H}_4 \cup \mathcal{A}_{2k-4}$ where $\mathcal{A}_{2k-4} \in \Omega_{2k-4}$.
Iterating our argumentation, we get a sequence of curves $\mathcal{A}_{2i} \in \Omega_{2i}$, $1 \leq i \leq k-1$, such that the deformation of $\mathcal{A}_{2i} \cup \mathcal{H}_{2k-2i}$ leads to the curve $\mathcal{H}_{2k}$. We also define a set of arcs I_{j+1},$2 \leq j \leq (2k-1)$, with the property that the curve $\mathcal{H}_j$ is of type $\mathcal{H}^0$ relatively to $I_{j+1} \approx \mathbf{R}L_{j+1}$. Arcs I_j, $3 \leq j \leq 2k$, belong to the non-empty oval of $\mathcal{A}_{2k-2}$; the set of arcs $\cup_j I_j$, $2k-2i+1 \leq j \leq 2k$ is mapped to a set of arcs of $\mathcal{A}_{2i}$, $1 \leq i \leq k-1$.

At the final step, we get $\mathcal{H}_{2k}$ from the deformation of $\mathcal{A}_2 \cup \mathcal{H}_{2k-2}$.

In the construction of the sequence of curves $\mathcal{A}_{2i}$, we have to take into account the way $\mathcal{H}_{2k-2i}$ intersects $\mathcal{A}_{2i}$. Curves $\mathcal{A}_{2i}$ may intersect curves $\mathcal{H}_{2k-2i}$ in only two different ways: the non-empty oval of one curve contains all inner ovals of the other curve or it contains none of them. Let us first consider this last case.

1) Assume that the non-empty oval of $\mathcal{A}_{2i}$ does not contain ovals of the non-empty oval of $\mathcal{H}_{2k-2i}$. The curve $\mathcal{H}_{2k}$ results from the deformation of the union $\mathcal{H}_{2k-2i} \cup \mathcal{A}_{2i}$. In this case, any arc I_j, $2k-2i+1 \leq j \leq 2k$, may be identified with the same line $\mathbf{R}L_{2k-2i+1}$. Thus, from step of the construction of $\mathcal{H}_{2k-2i+1}$ the method proposed for the construction of $\mathcal{H}_{2k}$ is analogous to the Harnack's construction.

Since arcs I_j, $2k-2i+1 \leq j \leq 2k$, are distinct, one can say equivalently -identifying I_j with lines $\mathbf{R}L_j$- that there exists $p \in L_{2k-2i+1} \cap L_j$, $2k-2i+1 \leq j \leq m+1$, and a rotation of center p and angle θ which maps $\mathbf{R}L_i$ to $\mathbf{R}L_{i+1}$. For simplicity, we choose the sequence of arcs I_j, $2k-2i+1 \leq j \leq 2k$ as follows. The arc I_{j+1} intersects the arcs I_j and I_{j+2}; I_{2k} is extreme.

2)Let us now assume that the non-empty oval of the curve $\mathcal{H}_2$ contains all inner ovals $\mathcal{A}_{2k-2}$. In the case **1)**, we have chosen arcs I_j, $3 \leq j \leq 2k$ such that an arc intersects at most an other arc in one point. We need here one more assumption on the choice of arcs I_j of the non-empty oval of $\mathcal{A}_{2k-2}$.
Assume that for any $i, 1 \leq i < k$, the non-empty oval of the curve $\mathcal{A}_{2k-2i}$ does not contain inner ovals of $\mathcal{H}_{2i}$. Let I_3 and I_4 be the two extreme arcs of $\mathcal{A}_{2k-2}$. Then, we get $\mathcal{H}_4$ and a curve $\mathcal{A}_{2k-4} \in \Omega_{2k-4}$ with the property that $\mathcal{H}_{2k}$ results from the deformation of $\mathcal{H}_4 \cup \mathcal{A}_{2k-4}$. We iterate this process. Consider the union I_j, $3 \leq j \leq 2k$, of arcs of $\mathcal{A}_{2k-2}$ as follows: I_3 and I_4 are extreme arcs of $\mathcal{A}_{2k-2}$ Up to rigid isotopy, arcs I_j, $j > 5$, are arcs of the non-empty oval empty of $\mathcal{A}_{2k-4}$. We choose I_5 and I_6 such that I_5 intersects I_3 in one point, I_6 intersects I_4 in one point. In such a way, they are mapped to extreme arcs of $\mathcal{A}_{2k-4}$. By induction, we choose I_{3+2l}, I_{4+2l}, $1 \leq l \leq (k-1)$ such that I_{3+2l} intersects $I_{3+2(l-1)}$ in one point, I_{4+2l} intersects $I_{4+2(l-1)}$ in one point; and are extreme arcs of $\mathcal{A}_{2k-2(l+1)}$. At the end of the process, we get $\mathcal{H}_{2k}$ from the deformation of $\mathcal{A}_2 \cup \mathcal{H}_{2k-2}$ where $\mathcal{A}_2 = \mathcal{H}_2$. (Identifying arcs I_j with lines, $\mathbf{R}L_{2k-1} \approx I_{2k-1}$, $\mathbf{R}L_{2k} \approx I_{2k}$ the curve $\mathcal{A}_2 = \mathcal{H}_2$, as L-curve, results from the perturbation of the union $L_{2k-1} \cup L_{2k}$.)

It is an easy property of the proposed construction that the oval of $\mathcal{A}_2 = \mathcal{H}_2$ contains besides inner ovals of $\mathcal{H}_{2k-2}$ inner ovals of $\mathcal{H}_{2k}$ in its inner component. Iterating this process, we get $\mathcal{H}_{2k-2i}$, $i < k$, from $\mathcal{H}_{2k-2i-2} \cup \mathcal{A}_2$ where the oval of $\mathcal{A}_2 = \mathcal{H}_2$ contains, besides inner ovals of $\mathcal{H}_{2k-2i-2}$ inner ovals of $\mathcal{H}_{2k-2i}$ in its inner component. (Identifying arcs I_j with lines, the curve $\mathcal{A}_2 = \mathcal{H}_2$, as L-curve, results from the perturbation of two lines $\mathbf{R}L_{2k-2i-1} \approx I_{2k-2i-1}$, $\mathbf{R}L_{2k-2i} \approx I_{2k-2i}$ in general position.)

If there exists $j \leq k$ such that the non-empty oval of $\mathcal{A}_{2k-2j}$ does not contain inner ovals of $\mathcal{H}_{2j}$, then from the step when the deformation $\mathcal{H}_{2j} \cup \mathcal{A}_{2k-2j}$ leads to $\mathcal{H}_{2k}$ to the final step - the deformation of $\mathcal{A}_2 \cup \mathcal{H}_{2k-2}$ leads to $\mathcal{H}_{2k}$ - , we are in the case **1)**.

In the previous argumentation, we have assumed that the non-empty oval of $\mathbf{R}\mathcal{A}_{2k-2}$ is the union of an arc which does not contain points of $\mathbf{R}\mathcal{A}_{2k-2} \cap \mathbf{R}\mathcal{H}_2$. with a set of arcs I_j, $3 \leq j \leq 2k$ which contains points of $\mathbf{R}\mathcal{A}_{2k-2} \cap \mathbf{R}\mathcal{H}_2$ From identification of these $2k-2$ arcs I_j with lines $\mathbf{R}L_j$, $3 \leq j \leq 2k$, we have described components of $\mathbf{R}\mathcal{A}_{2k-2}$ as an arrangement of pieces of these $2k-2$ lines. The degree of $\mathcal{A}_{2k-2}$ is $2k-2$. Hence, its real components may not result from more than $2k-2$ lines. Moreover, since $\mathcal{A}_{2k-2}$ is an M-curve, its real components may not result from less than $2k-2$ lines. Our initial assumption is therefore always possible. Moreover, one can assume that $\mathcal{H}_{2k}$ results from the deformation of $\mathcal{A}_{2k-2} \cup \mathcal{H}_2$ where $\mathcal{A}_{2k-2} \in \Omega_{2k-2}$. and $\mathbf{R}\mathcal{A}_{2k-2}$ is entirely covered by the set of arcs $\cup_{j=3}^{2k} I_j$. Hence, since $\mathcal{H}_2$ is, up to rigid isotopy, an L-curve; we get Theorem 2.28 of Chapter 2 for Harnack curves $\mathcal{H}_{2k}$ of even degree.

II) Harnack Curves of odd degree $\mathcal{H}_{2k+1}$

According to Proposition 2.17 of Chapter 2, up to regular modification of its Harnack polynomial B_{2k+1}, one can assume that $\mathcal{H}_{2k+1}$ results from the deformation of the union $\mathcal{A}_{2k-2} \cup \mathcal{H}_3$, where $\mathcal{A}_{2k-2} \in \Omega_{2k-2}$ and $\mathcal{H}_3$ is the Harnack curve of degree 3. Curves $\mathcal{H}_3$ are rigidly isotopic. One can consider $\mathcal{H}_3$ as an L-curve i.e as the result of the perturbation of three real lines L_0, L_1,L_2 in general position. According to remark 2.16 of Chapter 2, we shall assume lines L_0,L_1, L_2 chosen as follows. Outside a neighborhood $\mathcal{B}$ of singular points $L_0 \cup L_1 \cup L_2$, there exists a rigid isotopy j_t of $\mathbf{R}P^2$ which pushes $\cup_{i=0}^{2} \mathbf{R}L_i \backslash \mathcal{B}$ to $\mathbf{R}\mathcal{H}_3 \backslash \mathcal{B}$ such that $\mathbf{R}\mathcal{H}_3$ intersects $\mathbf{R}\mathcal{A}_{2k-2}$ only its part $j_1(\cup_{i=0}^{2} \mathbf{R}L_i \backslash \mathcal{B}) \backslash j_1(\mathbf{R}L_0)$. Consider the deformation $\mathcal{H}_2 \cup \mathcal{A}_{2k-2}$ where $\mathcal{H}_2$ is an L-curve which results from the perturbation of the two real lines L_1,L_2 in general position. Combining this construction of curves $\mathcal{H}_{2k+1}$ and our previous study of curves of even degree, we obtain the Theorem 2.28 of Chapter 2 for curves $\mathcal{H}_{2k+1}$. Q.E.D

To prove the rigid isotopy Theorem 2.1 of Chapter 2 it is sufficient to note that according to Proposition 2.25 of Chapter 2 and the Theorem 2.28 of Chapter 2 and its proof, any curve $\mathcal{H}_m$ is of type $\mathcal{H}^0$. Thus, the rigid isotopy Theorem 2.1 of Chapter 2 is a straightforward consequence of the Theorem 2.2 of Chapter 2.

However, for clarity, we propose to prove the Theorem 2.30 of Chapter 2.

THEOREM 2.30. *Up to rigid isotopy, any curve $\mathcal{H}_m$, $m \geq 1$, results from the successive classical small deformation of the union $L_{i+1} \cup \mathcal{H}_i$, $1 \leq i \leq n-1$, where $\mathcal{H}_i$ is a Harnack curve of degree i, L_{i+1} is a projective line and the curve $\mathcal{H}_i$ is of type $\mathcal{H}^0$ relatively to L_{i+1}.*

Let us give in Proposition 2.31 of Chapter 2 an other formulation of the Theorem 2.30 of Chapter 2.

PROPOSITION 2.31. *Let $\mathcal{H}_m$ be a Harnack curve of degree m and $B_m(x_0, x_1, x_2)$ its polynomial.*
Denote by $b_m(x, y)$ the affine polynomial associated to $B_m(x_0, x_1, x_2)$,

$$B_m(x_0, x_1, x_2) = x_0^m.b_m(x_1/x_0, x_2/x_0)$$

Denote by $b_i(x, y)$ the truncation of $b_m(x, y)$ on the monomials $x^\alpha.y^\beta$ with $0 \leq \alpha + \beta \leq i$ and by $B_i(x_0, x_1, x_2) = x_0^i.b_i(x_1/x_0, x_2/x_0)$ the homogeneous polynomial associated to b_i. Then, up to linear change of projective coordinates, and up to slightly perturb coefficients of $B_m(x_0, x_1, x_2)$ without changing either order or topological structure of $\mathcal{H}_m$, we may assume that any polynomial B_i is of type $\mathcal{H}^0$ relatively to $x_0 = 0$.

proof: According to Theorem 2.28 of Chapter 2, up to rigid isotopy, there exists a sequence of lines $\mathcal{H}_1 = L_1$, L_2, ..., L_{m+1} such that $\mathcal{H}_i$ is of type $\mathcal{H}^0$ relatively to L_{i+1} and $\mathcal{H}_{i+1}$ results from the deformation of $\mathcal{H}_i \cup L_{i+1}$. The sequence of lines L_i may be ordered in three different ways described in (1), (2) and (3). To prove to Theorem 2.30, it is sufficient to prove that the description (1) is rigidly isotopic to the description (2).
Without loss of generality, one can assume that points $\mathcal{H}_i \cap L_{i+1}$ do not belong to the line at infinity of $\mathbf{R}P^2$. Any affine line $l_j = \mathbf{R}^2 \cap L_j$ is divided into two

halves l_j^+ l_j^- with common point p. (The rotation of center p maps a positive (resp, negative) half of one line to a positive (resp, negative) half of an other line.) Consider the half plane $\mathbf{R}^+$ (resp, $\mathbf{R}^-$) which contains positive (resp, negative) halves lines. Let H be the common line of $\mathbf{R}^+$ and $\mathbf{R}^-$ (where $H \cap L_j = p$, for any L_j, $1 \leq j \leq m$). The symmetry s_p of center p exchanges halves of lines. Let us denote s_H the orthogonal reflection with respect to H. Assume, as described in (2), that there exists $p \in L_2 \cap L_j$, $3 \leq j \leq m+1$, such that:
- for i, $2 \leq i \leq 2.[(m+1)/2] - 2$ the rotation of center p and angle θ maps $\mathbf{R}L_i$ to $\mathbf{R}L_{i+2}$.
- for i, $1 \leq i \leq 2.[(m+1)/2] - 2$ the rotation of center p and angle $-\theta$ maps $\mathbf{R}L_{i+1}$ to $\mathbf{R}L_{i+3}$ It follows from the remark 2.16 that without loss of generality, one can assume that $\mathcal{H}_i$ intersects L_{i+1} in its half l_{i+1}^+. Choose the lines L_j such that the symmetry s_H exchanges lines L_{2l-1} and L_{2l}, $s_H(L_{2l-1}) = L_{2l}$,$s_H(L_{2l}) = L_{2l-1}$ According to remark 2.16 of Chapter 2, the rigid isotopy is trivially preserved under the action of s_H.

Let us now choose lines L_j such that the rotation of center p and angle $\pm\frac{\theta}{2}$ maps $s_H(L_{2l})$ to $L_{2l\pm1}$. Obviously, the description of $\mathcal{H}_m$ which results from this arrangement of lines L_j and the one where $s_H(L_{2l-1}) = L_{2l}$,$s_H(L_{2l}) = L_{2l-1}$ are rigidly isotopic. The rigid isotopy is preserved by the symmetry s_p which maps the intersection of $\mathcal{H}_i$ with L_{i+1} (namely, with l_{i+1}^+) to a set of the half l_{i+1}^-. The rigid isotopy is also preserved when consider $s_H(L_{2j}) = L'_{2j}$ and L_{2j-1} i.e $l_{2j}^{'+}$ and l_{2j-1}^+ instead of halves lines l_{2j-1}^-, l_{2j}^- In this way, we get that the description (2) is rigidly isotopic to the description (1). This proves the Theorem 2.30 of Chapter 2. Q.E.D

Harnack curves $\mathcal{H}_m$ constructed from the Harnack's method are rigidly isotopic. It follows that the Theorem 2.30 of Chapter 2 implies the Rigid Isotopy Classification Theorem 2.1 of Chapter 2.

Appendix

Curves $\mathcal{H}_m$ of type $\mathcal{H}^0$ may result from the Harnack's inductive construction ([**11**], [**18**]) as follows.

Given a projective line L, each curve $\mathcal{H}_{i+1}$, $1 \leq i \leq m-1$, results from the classical small perturbation of the union $\mathcal{H}_i \cup L$ where $\mathcal{H}_i$ is of type $\mathcal{H}^0$ relatively to L; $\mathcal{H}_1$ is a projective line which intersects L in one point.

We take the auxiliary curve $\mathcal{C}_{i+1}$ which perturb the union $\mathcal{H}_i \cup L$ to be a union of $i+1$ lines.

Consider L as the line at infinity of $\mathbf{R}P^2$. From the Harnack's method [**11**] initiated with L, one can obtain isotopy of curves different from $\mathcal{H}_m$ and curves $\mathcal{H}_m$ which are not of type $\mathcal{H}^0$ relatively to L. The isotopy type of curves (projective and affine) which result from the Harnack's method depends on the choice of auxiliary curves.

Let us prove in Theorem 2.32 of Chapter 2 that isotopy implies also rigid isotopy for Harnack curves of type $\mathcal{H}^0$ obtained from this method. For curves of degree $m \leq 6$, the Theorem is obvious.

THEOREM 2.32. *Any two Harnack curves $\mathcal{H}_m$ of type $\mathcal{H}^0$ constructed from the Harnack's method are rigidly isotopic.*

proof: We shall proceed by ascending induction on the degree.
The Theorem 2.32 of Chapter 2 is trivial for Harnack curves of degree $\mathcal{H}_i$, $i \leq 6$. Let us start our induction with Harnack curves of degree 1. Let L_1 and L_2 be two projective lines, and $\mathcal{H}_1^1$, $\mathcal{H}_1^2$ be Harnack curves of degree 1. Consider the classical small perturbation of $\mathcal{H}_1^1 \cup L_1$ and $\mathcal{H}_1^2 \cup L_2$. Denote by $\mathcal{H}_2^1$,(resp, $\mathcal{H}_2^2$), the Harnack curve of degree 2 which results from the classical deformation of $\mathcal{H}_1^1 \cup L_1$ (resp, $\mathcal{H}_1^2 \cup L_2$).

Any two real projective lines are rigidly isotopic. Let h_t be the rigid isotopy $[0,1] \rightarrow \mathbf{R}\mathcal{C}_1 \backslash \mathbf{R}\mathcal{D}_1$,$h_0(\mathbf{R}L_1) = (\mathbf{R}L_1)$, $h_1(\mathbf{R}L_1) = \mathbf{R}L_2$. Denote by $\mathbf{R}L_{t+1}$ the line $h_t(\mathbf{R}L_1)$. Any two projective lines intersect each other in one point. Hence, one can assume, without loss of generality that $\mathcal{H}_1 = \mathcal{H}_2 = L_0$. Along the rigid isotopy h_t, any line $\mathbf{R}L_{t+1} = h_t(\mathbf{R}L_1)$ intersects $\mathbf{R}\mathcal{H}_0$ in one point. Noticing that the classical perturbation of the union of two lines which intersect each other in one point leads to the Harnack curve $\mathcal{H}_2$, the path $h_t * id$: $[0,1] \rightarrow \mathbf{R}\mathcal{D}_2$ extends to a path $g_t : [0,1] \rightarrow \mathbf{R}\mathcal{C}_2 \backslash \mathbf{R}\mathcal{D}_2$, $g(0) = \mathbf{R}\mathcal{H}_2^1$, $g(1) = \mathbf{R}\mathcal{H}_2^2$ which is a rigid isotopy.

Given $\mathcal{H}_m^1$, (resp, $\mathcal{H}_m^2$), $i \geq 2$ the Harnack curve m deduced from the Harnack's method initiated with the line L_1, (resp, L_2). We shall prove that curves $\mathcal{H}_m^1$ and $\mathcal{H}_m^2$ are rigidly isotopic for arbitrary degree m.

Let us start with the following remark.

LEMMA 2.33. *Let L be a real projective line. Consider the Harnack's inductive construction of curves initiated with the line L. Any two Harnack curves $\mathcal{H}_m$ of type $\mathcal{H}^0$ obtained from the Harnack's method initiated with the line L are rigidly isotopic.*

proof: Indeed, consider the Harnack's construction of the curve $\mathcal{H}_m$ initiated with the line L. In this construction, any curve $\mathcal{H}_i$, $1 \leq i \leq m-1$, is of type $\mathcal{H}^0$ relatively to L. Consider L as the line at infinity of $\mathbf{R}P^2$. Given a Harnack curve $\mathcal{H}_i$, denote by h_i the affine corresponding curve. Denote by $\mathcal{C}_i$ the curve which deforms the union of L with the curve of degree $i-1$, $i \leq m$. Let the auxiliary curve of degree j $\mathcal{C}_j$ be the union of j lines of a pencil through a point. Noticing that the isotopy type of the affine curve h_i $(1 \leq i \leq m)$ depends exclusively on the position of $\mathcal{C}_j \cap L$,$1 \leq j \leq i$, we shall prove the Lemma 2.33 of Chapter 2. Let us detail our argumentation.

Let us prove that any two curves $\mathcal{H}_{i+1}$ and $\mathcal{H}'_{i+1}$ resulting from the classical perturbation of $\mathcal{H}_i \cup L$ directed to a union of $i+1$ lines $\mathcal{C}_{i+1}$, resp $\mathcal{C}'_{i+1}$ are rigidly isotopic.
Denote by x_0, B_i, C_{i+1},(resp, C'_{i+1}), B_{i+1} ,(resp,B'_{i+1}) the polynomial of L, $\mathcal{H}_i$, and $\mathcal{C}_{i+1}$, (resp $\mathcal{C}'_{i+1}$), $\mathcal{H}_{i+1}$, (resp, $\mathcal{H}'_{i+1}$).
In the construction of $\mathcal{H}_{i+1}$ of type $\mathcal{H}^0$ from $\mathcal{H}_i \cup L$, the curve $\mathcal{C}_{i+1}$ intersects L in $i+1$ distinct points lying
-for even $i+1$ (i.e construction of $\mathcal{H}_{2k}$ from $\mathcal{H}_{2k-1} \cup L$) in the component S_i of $\mathbf{R}L$ which is a boundary of the unique non-empty positive region $\{x \in \mathbf{R}^2 | b_i(x) > 0\}$ the non-empty region of $\mathbf{R}^2$ with boundary a part of the odd component of $\mathbf{R}\mathcal{H}_i$).
-for odd $i+1$, (i.e construction of $\mathcal{H}_{2k+1}$ from $\mathcal{H}_{2k} \cup L$) in the component S_i of

$\mathbf{R}L\backslash\mathbf{R}\mathcal{H}_i$ containing $\mathbf{R}L\cap\mathbf{R}\mathcal{H}_{i-1}$.
(It is also necessary that $\mathcal{C}_{i+1}$ does not intersect $\mathcal{H}_i\cup L$ in its singular points).
i.) If $\mathcal{C}_{i+1}\cap L=\mathcal{C}'_{i+1}\cap L$, one can vary continuously the direction of lines of $\mathcal{C}_{i+1}$ and in this way define a one parameter family $\mathcal{C}_{i+1,t;\ t\in[0,1]}$, $\mathcal{C}_{i+1,0}=\mathcal{C}_{i+1}$, $\mathcal{C}_{i+1,1}=\mathcal{C}'_{i+1}$, and thus a rigid isotopy $x_0.B_i+C_{i+1,t}$ from $\mathcal{H}_{i+1}$ to $\mathcal{H}'_{i+1}$.
ii.) Otherwise $\mathcal{C}_{i+1}\cap L\neq\mathcal{C}'_{i+1}\cap L$. In the construction of $\mathcal{H}_{i+1}$ of type $\mathcal{H}^0$ from classical deformation of $\mathcal{H}_i\cup L$, any auxiliary curve which deforms the union $\mathcal{H}_i\cup L$ intersects $\mathbf{R}L$ in $i+1$ points lying in a connected part of S_i, $S_i\subset\mathbf{R}L$. Given $\mathcal{C}_{i+1}$, denote $\mathcal{C}_{i+1}\cap L$ by I_{i+1} and let I'_{i+1} be a set of $i+1$ real points of S_i. Let us distinguish the cases $i+1$ even and odd.
-In case $i+1=2k$ even, (i.e construction of $\mathcal{H}_{2k}$ from $\mathcal{H}_{2k-1}\cup L$) $S_i=S_{2k-1}$ is connected. According to the previous study, one can assume that lines of $\mathcal{C}_{2k}$ and $\mathcal{C}'_{2k}$ have the same direction. It is not hard to see that there exists a rigid isotopy $B_{2k,t}$ with $t\in[0,1]$ of $B_{2k,0}=x_0.B_{2k-1}+C_{2k}$, $B_{2k,1}=x_0.B_{2k-1}+C'_{2k}$ such that $\mathcal{C}'_{2k}(x_0,x_1,x_2)\cap L=I'_{2k}$.
-The case $i+1=2k+1$ odd (i.e construction of $\mathcal{H}_{2k+1}$ from $\mathcal{H}_{2k}\cup L$) differs from the case i odd in the sense that the set $S_i=S_{2k}$ has two connected parts. These two connected parts may be defined using a projection to L in a direction perpendicular to L as follows. The non-empty oval of $\mathbf{R}\mathcal{H}_{2k}$ intersects the line L in $2k$ points. They belong to a segment of L with extremities 2 points of $\mathbf{R}\mathcal{H}_{2k}\cap L$. Using a pencil of lines in a direction (for example in a direction perpendicular) to L, project the non-empty oval of $\mathbf{R}\mathcal{H}_{2k}$ to a segment S of L. The set S_{2k} consists of the two segments $S^1_{2k}=[a_1,b_1[$ and $S^2_{2k}=[a_2,b_2[$ with extremities an extremity of S and a point of $\mathbf{R}\mathcal{H}_{2k}\cap L$ (this last one is the open extremity of the segment). Changing the direction of the projection, one defines two other segments. (The idea in our proof is to find a way to join this two segments. It may be done by changing the direction of the projection.)
Let $\mathcal{H}_{2k+1}$, such that $\mathcal{H}_{2k+1}\cap S^2_{2k}=\emptyset$ and $\mathcal{H}'_{2k+1}\cap S^2_{2k}=\not\emptyset$. To prove that the curves $\mathcal{H}_{2k+1}$ and $\mathcal{H}'_{2k+1}$ are rigidly isotopic, it is sufficient to prove that there exists a continuous path $\mathcal{H}_{2k+1,t}$ $t\in[0,1]$, $\mathcal{H}_{2k+1,0}=\mathcal{H}_{2k+1}$, $\mathcal{H}_{2k+1,1}$ with the following properties:
-for any $t\in[0,1]$ $\mathcal{H}_{2k+1,t}$ is a Harnack curve of degree $2k+1$
-as t varies from 0 to 1 a line of $\mathcal{H}_{2k+1}$ (i.e of $\mathcal{C}_{2k}$) intersecting S^1_{2k+1} is deformed to a line of $(\mathcal{H}'_{2k+1,1})$ intersecting S^2_{2k}.

(Recall that if $\mathcal{C}_{2k+1}\cap L=\mathcal{C}'_{2k+1}\cap L$ there exists a rigid of isotopy $x_0.B_{2k}+C_{2k+1,t}$, $t\in[0,1]$, from $\mathcal{H}_{2k+1}$ to $\mathcal{H}'_{2k+1}$. Hence, we shall consider $\mathcal{C}_{2k+1}$ and $\mathcal{C}'_{2k+1}$ up to such rigid isotopy.)

Let γ be the non-empty arc (of the non-empty oval) of $\mathcal{H}_{2k}$ which intersects L in two points $a_1\in S^1_{2k}=[a_1,b_1[$ $a_2\in S^2_{2k}=[a_2,b_2[$. Let $p\in\gamma$, consider a path γ_t, $\gamma_0=p$ which moves p along γ.

Let $p_1\in S^1_{2k}$, and $L_1=(pp_1)$ be a line through p. Consider the pencil of lines through the point p which intersect γ. Move continuously the point p along the path $\gamma_t\subset\gamma$ and in this way lines $(p_1\gamma_t)$ of the pencil of lines through p_1. The projection of p (in a direction perpendicular to L) to L belongs to S^1_{2k}. It is not hard to see that one can choose the path γ_t such that, as p moves along γ_t, the projection γ_t to L moves from S^1_{2k} to $p_2\in S^2_{2k}$.

In this way, considering together pencil of lines through any point $p_t := \gamma_t$ and pencils through the point $p_1 \in S^1_{2k}$ and pencils through the point $p_2 \in S^2_{2k}$ it follows that there exists a rigid isotopy of $\mathcal{H}_{2k+1}$ from $\mathcal{H}'_{2k+1}$ which maps a line of $\mathcal{C}_{2k+1}$ which intersects S^1_{2k} to a line of a curve $\mathcal{C}'_{2k+1}$ which intersects S^2_{2k}.

It concludes the proof of the Lemma 2.33 of Chapter 2. Q.E.D

It follows from the Lemma 2.33 of Chapter 2 that when one considers Harnack's construction of curves $\mathcal{H}_m$ up to rigid isotopy, we may restrict our study to classical deformation of $\mathcal{H}_i \cup L$, $1 \le i \le (m-1)$, directed to a union of $i+1$ lines of a pencil through a point p chosen outside L and $\mathcal{H}_i$.

Denote $\mathcal{C}_{i+1}$ the union of lines which deform the union of L with the curve of degree i, $i \le m$. The isotopy type of curves of degree m obtained from the Harnack's recursive method depends on the relative position of intersection points of $\mathcal{C}_{i+1}$ with the line L.

Consider the Harnack's construction of curves $\mathcal{H}_i$, $1 \le i \le m$. Denote by B_i the polynomial of $\mathcal{H}_i$. Consider L as the line at infinity, and denote by b_i the affine polynomial associated to B_i. Curves $\mathcal{H}_i$ constructed from the Harnack's method are of type $\mathcal{H}^0$ relatively to L.

Let $\mathcal{H}_5$ be the Harnack curve of degree 5. There exists a unique positive region $\{x \in \mathbf{R}^2 | b_5(x) > 0\}$ with a segment S_5 of the line $\mathbf{R}L$ on its boundary which contains an oval of $\mathcal{H}_5$. To get $\mathcal{H}_6$ from $\mathcal{H}_5 \cup L$, it is necessary that the intersection of $\mathcal{C}_6$ with L consists of 6 points lying in S_5. The non-empty oval of $\mathcal{H}_6$ intersects L in 6 points. Denote S_6 the connected part of the line $\mathbf{R}L$ which contains the intersection points of $\mathcal{H}_6$ with L. To get $\mathcal{H}_7$ from $\mathcal{H}_6 \cup L$, it is necessary that the intersection of $\mathcal{C}_7$ with L consists of 7 points lying in $L \backslash S_6$ and that it intersects $\mathcal{H}_6$ in its non-empty real component. Iterating this process, (set $2k+1$ (resp, $2k$) instead of 5 (resp, 6)) we define a sequence S_i of connected components of $\mathbf{R}L$ such that $S_i \subset \mathcal{C}_{i+1} \cap L$.

In other words, we define relative position of lines $\mathcal{C}_{i+1}$ as follows. We take $\mathcal{C}_m$ to be a union of m lines which intersect L in m distinct points lying, for even m in the component of $\mathbf{R}L$ which is a boundary of the unique non-empty positive region $\{x \in \mathbf{R}^2 | b_{m-1}(x) > 0\}$, for odd m in the component of $\mathbf{R}L \backslash \mathbf{R}\mathcal{H}_{m-1}$ containing $\mathbf{R}L \cap \mathbf{R}\mathcal{H}_{m-2}$. (It is also necessary to choose $\mathcal{C}_m$ such that it does not intersect $\mathcal{H}_{m-1} \cup L$ in its singular points).

Denote by B^1_i (resp, B^2_i) the polynomial of $\mathcal{H}^1_i$ (resp, $\mathcal{H}^2_i$) and by C^1_{i+1} (resp, C^2_{i+1}) the union of $i+1$ lines which deform $\mathcal{H}^1_i \cup L_1$ (resp, $\mathcal{H}^2_i \cup L_1$) to $\mathcal{H}^1_{i+1}$ (resp, $\mathcal{H}^2_{i+1}$).

To prove Theorem 2.32 of Chapter 2, is sufficient to construct for i, $1 \le i \le (m-1)$ a continuous one parameter family with parameter $t \in [0,1]$ of curves $\mathcal{C}^{1+t}_{i+1}$ such that as t varies from 0 to 1 the relative position of lines $\mathcal{C}^{1+t}_{i+1}$ remains the same.

The rigid isotopy h_t is continuous. Therefore, the function which associates to any line $h_t(\mathbf{R}L_1)$ its normal vector $\vec{n}_t$ and its tangent vector $\vec{v}_t$ is also continuous. It follows from the continuity of h_t, that one can choose lines $\mathcal{C}^{1+t}_{i+1}$ such that the relative position of intersections $\mathcal{C}^{1+t}_{i+1} \cap L^{1+t}$, and also the direction of lines of $\mathcal{C}^{1+t}_{i+1}$ relatively to $(\vec{v}_t, \vec{n}_t)$ remains the same as t varies from 0 to 1. In such a way, for any $i \le (m-2)$ we construct by induction curves $\mathcal{H}^t_{i+2}$, $t \in [0,1]$, of degree $i+2$ rigidly

isotopic. Hence, for any $i \leq (m-2)$, curves $\mathcal{H}^1_{i+2}$ and $\mathcal{H}^2_{i+2}$ are rigidly isotopic. In particular, curves $\mathcal{H}^1_m$ and $\mathcal{H}^2_m$ are rigidly isotopic. Q.E.D

3. Harnack Curves from a Complex viewpoint

In this section, we shall first construct particular deformations of Harnack polynomials. Then, we shall deduce from the properties of these deformations a characterization of the complex set point of Harnack curves in $\mathbf{C}P^2$.
We shall divide this section into two subsections. Denote $\mathcal{H}_m$ the Harnack curve of degree m defined up to isotopy of real points set. In the first section, we define (in proposition 3.6 of Chapter 2) deformation of any Harnack curve $\mathcal{H}_m$ to a singular curve of which singular points are critical points index 1 of the Harnack curve $\mathcal{H}_m$. At first, we characterize in proposition 3.1 of Chapter 2 such deformation for Harnack curves obtained via the pathchworking method and then generalize it in proposition 3.6 of Chapter 2 to any Harnack curve. Considering patchworking method seems at first glance of less interest at this time of our proof of the Rokhlin's Conjecture. Nonetheless, it will be useful in the second part (Perestroika Theory on Harnack curves). In the next section, we deduce from the proposition 3.6 of Chapter 2 a description of $(\mathbf{C}P^2, \mathbf{C}\mathcal{H}_m)$ up to conj-equivariant isotopy and give the main result (Theorem 3.9 of Chapter 2) of this chapter.

Deformation of Harnack Curves. Given a non-singular curve $\mathcal{P}_m$ in $\mathbf{C}P^2$, call *simple deformation* $\mathcal{P}_{m;t}$ of the curve $\mathcal{P}_m$ a path

$$[0,1] \to \mathbf{R}\mathcal{C}_m$$
$$t \to \mathcal{P}_{m;t}$$

with the following properties:

(1) For any $0 \leq t < 1$, $\mathcal{P}_{m;t}$ is a smooth curve. If $\mathcal{P}_{m;1}$ is singular, then it is irreducible and any of its singular points is a real crossing which is critical point with positive critical value of the affine Harnack polynomial. Denote S the set of singular points of$\mathcal{P}_{m;1}$.
(2) For $\epsilon > 0$, let $\mathcal{D}_\epsilon = \cup_{a\in S} D_(a, \epsilon)$ be the union of disc $D(a, \epsilon)$ (in the Fubini-Study metric) of center a and radius ϵ taken over crossings a of $\mathcal{P}_{m;1}$.

There exists $\epsilon_0 > 0$, such that: for any curve $\mathcal{P}_{m,t}$, $t \in]0,1[$, $\mathbf{C}P_{m,t}$ lies in an ϵ_0-tubular N neighborhood of $\mathbf{C}P_{m,1} \backslash \{a \in S\}$.
$\mathbf{C}P_{m,t}$ can be extented to the image of a smooth section of the tubular fibration $N \to \mathbf{C}P_{m,1} \backslash \{a \in S\}$.

Under two distinct simple deformations $\mathcal{P}_{m;t}$ and $\tilde{\mathcal{P}_{m;t}}$ of $\mathcal{P}_m$, the number of real singular points of $\mathcal{P}_{m;1}$ and $\tilde{\mathcal{P}_{m;1}}$ may be distinct.

Among simple deformations we shall distinguish deformations which lead to a curve $\mathcal{P}_{m;1}$ with the maximal number of real singular points a curve $\mathcal{P}_{m;1}$ may have. Denote by α this number. We shall call *maximal simple deformation* $\mathcal{P}_{m;t}$, $t \in [0,1]$, of the curve $\mathcal{P}_m$ a simple deformation $\mathcal{P}_{m;t}$ such that the curve $\mathcal{P}_{m;1}$ has α real singular points.

In proposition 3.6 of Chapter 2, we define some deformation of $\mathbf{R}\mathcal{H}_m$,

$$[0,1] \to \mathbf{R}\mathcal{C}_m$$
$$t \to \mathbf{R}\mathcal{H}_{m;t}$$

to the real points set $\mathbf{R}\mathcal{A}$ of a singular curve $\mathcal{A}$ of degree m of which singular points are critical points of index 1 of the Harnack curve $\mathcal{H}_m$. At first, we characterize in proposition 3.1 of Chapter 2 such deformation for Harnack curves obtained via the pathchworking method and then generalize it in proposition 3.6 of Chapter 2 to any Harnack curve.

Recall that given a Harnack polynomial B_{2k} of degree $2k$ type $\mathcal{H}^0$ and associated affine polynomial b_{2k}, S'_{2k} denotes the subset of critical points (x_0, y_0) of positive critical value c_0 with the property that as c increases from $c_0 - \epsilon$ to $c_0 + \epsilon$ the number of real connected components of $M_c = \{((x,y) \in \mathbf{R}^2 | b_{2k} > c\}$ bounding the line at infinity increases.

Maximal Deformation of T-Harnack curves. Let us in Proposition 3.1 of Chapter 2 characterize maximal simple deformation of T-Harnack curves. It is obious that the Proposition 3.1 of Chapter 2 may be generalized to any Harnack curve. The general statement 3.6 of Chapter 2 may be proven independently of Proposition 3.6 of Chapter 2. For sake of clarity, we present here deformation of T-Harnack curves.

PROPOSITION 3.1. (1) *Along any simple deformation of a T-Harnack curve of odd degree $2k+1$ curves are smooth.*

(2) *Given $\mathcal{H}_{2k}$ a T-Harnack curve of even degree $2k$, any maximal simple deformation of $\mathcal{H}_{2k}$*

$$[0,1] \to \mathbf{C}\mathcal{C}_m$$
$$t \to \mathcal{H}_{m;t}$$

is such that $\mathcal{H}_{2k;1}$ has $2k-3$ crossings.
Let S be the set of singular points of $\mathcal{H}_{2k;1}$. Then, there exists a Harnack polynomial B_{2k} of degree $2k$ and type $\mathcal{H}^0$ of which the set of critical points contains the set S.
These points are critical points with positive critical value. Besides, $k-1$ of these points are points of S'_{2k}.

As explained in section 1 of Chapter 0, one can give via the Patchworking method an inductive construction of Harnack curves called T-inductive contruction.

Recall that in the T-inductive construction of Harnack polynomials $\vec{a}_m, t_m$ denotes the pair constituted by a vector $\vec{a}_m$ and a real t_m such that for any $t \in]0, t_m[$, $\tilde{X}_{m;\vec{a}_m,t} = \sum_{(i,j)\ vertices\ of\ T_m} \epsilon_{i,j} a_{i,j} x_1^i x_2^j x_0^{m-i-j} t^{\nu(i,j)}$ is a Harnack polynomial of degree m.
Given $\vec{a}_m, t_m$, $\vec{a}_{m+1} = (\vec{a}_m, \vec{c}_{m+1})$.

REMARK 3.2. (1) There exists $\vec{c}_{2k+1}$ such that $t_{2k+1} = t_{2k}$.

(2) There exists $\vec{c}_{2k}$ and $\tau < t_{2k-1}$ such that for $t = \tau$ the curve given by $\tilde{X}_{2k;\vec{a}_{2k},\tau}$ has $2k-3$ singular points which are collinear crossings.

proof:

Consider a T-Harnack curve of degree m.
For $m=1$ and $m=2$, the proposition 3.1 of Chapter 2 is trivially verified. For $m>2$, it is based on the previous proposition 1.4 of Chapter 2.

We shall use the terminology introduced in the chapter 1 and denote $\tilde{X}_{m;t}$, $t \in]0, t_m[$ a T-Harnack polynomial of degree m. It is obvious that any T-Harnack polynomial is of type $\mathcal{H}^0$. In what follows, we shall deal with one-parameter polynomials $\tilde{X}_{m;t}$, $t \in [t_m, t_{m-1}[$.

When no confusion is possible, we shall denote B_m a regular Harnack polynomial of degree m of type $\mathcal{H}^0$ and S_m^-, respectively S_m^+, the set of its critical points of negative, respectively positive, critical value.

Let us recall a result of the patchworking theory.

LEMMA 3.3. *([**18**], p.190) Let a be a polynomial such that $a=0$ admits an ϵ-tubular neighborhood. If a set $U \subset \mathbf{C}P^2$ is compact and contains no singular points of $a=0$, then for any $\epsilon>0$ and any polyhedron $\Delta \supset \Delta(a)$ there exists $\alpha>0$ such that for any polynomial b with $\Delta(b) \subset \Delta$, $||b-a|| < \alpha$ and $b^{\Delta(a)} = a$ the truncation $b^{\Delta(a)}$ is ϵ-sufficient in U.*

We shall consider polynomials $\tilde{X}_{m+1;\tau}$ with the following properties. For any elementary triangle Δ of the triangulation of T_{m+1}:

(1) the truncation $\tilde{X}^{\Delta}_{m+1;\tau}$ is completely non-degenerate.
(2) $\tilde{X}^{\Delta}_{m+1;\tau}$ is ϵ-sufficient for $\tilde{X}_{m+1;\tau}$ in $\rho^{m+1}(\mathbf{R}_+\Delta^0 \times U_{\mathbf{C}^2})$

We shall say that such polynomial $\tilde{X}_{m+1;\tau}$ satisfies the *"good truncation properties"*. It is easy to see that any polynomial $\tilde{X}_{m+1;\tau}$ with good truncation properties is such that for any elementary triangle Δ, the truncation $\mathbf{R}\tilde{X}^{\Delta}_{m+1;\tau}$ is isotopic to $\mathbf{R}\mathcal{H}_{m+1}$ in the open $\rho^m(\mathbf{R}_+\Delta^0 \times U_{\mathbf{C}^2}) \subset (\mathbf{C}^*)^2$. (Obviously, any Harnack polynomial has good truncation properties.)

Let us give the main ideas which motivate our study and explain the method of our proof.

Motivation
The T-inductive construction of Harnack polynomials can be considered as a slightly modified version of Harnack's initial one.

Fix $t_0 \in]0, t_m[$, and denote by $B_m := \tilde{X}_{m;t_0} = \sum_{(i,j)\ vertices\ of\ T_m} \epsilon_{i,j} a_{i,j} x_1^i x_2^j x_0^{m-i-j} t_0^{\nu(i,j)}$ the corresponding T-Harnack polynomial.

The T-Harnack polynomial $\tilde{X}_{m+1} = \sum_{(i,j)\ vertices\ of\ T_{m+1}} \epsilon_{i,j} a_{i,j} x_1^i x_2^j x_0^{m+1-i-j} t_0^{\nu(i,j)}$ of degree $m+1$ may be deduced by the formula: $\tilde{X}_{m+1;t} = x_0.B_m + t_0.C_{m+1}$ where x_0 is a line, the curve given by C_{m+1} is the union of $m+1$ parallels lines which intersect $x_0=0$ and do not pass through the singular points of the curve $\mathcal{A}_{m+1}$ of degree $m+1$ with polynomial $x_0.B_m$. For t sufficiently small, in particular for $t_0 < t_{m+1}$, $\tilde{X}_{m+1;t} = x_0.B_m + t.C_{m+1}$ is a Harnack polynomial B_{m+1}.

Therefore, outside $\mathbf{R}C_{m+1}$ the curve $\mathbf{R}\tilde{X}_{m+1;t_0}$ is a level curve of the function $\frac{x_0.B_m}{C_{m+1}}$. On $\mathbf{R}\mathcal{A}_{m+1} \backslash \mathbf{R}C_{m+1}$, this level curve has critical points only at the singular

points of $\mathbf{R}\mathcal{A}_{m+1}$. These points are non-degenerate singular points. Hence, the behavior of $\mathbf{R}\tilde{X}_{m+1;t_0}$ outside $\mathbf{R}C_{m+1}$ is described by the implicit function theorem and Morse Lemma. In particular, we have the following description:

(1) Let $\{a_1, ..., a_m\}$ be the set of crossings of $x_0.B_m = 0$. Denote $D(a_i, \epsilon)$ a small neighborhood of radius ϵ around a_i in $\mathbf{R}P^2$.

Denote $\mathcal{D}_\epsilon = \cup_{i=1}^m D(a_i, \epsilon)$ the neighborhood of the set of singular points of $\mathbf{R}\mathcal{A}_{m+1}$ in $\mathbf{R}P^2$. Let N be a tubular neighborhood of $\mathbf{R}\mathcal{A}_{m+1}\backslash\mathcal{D}_\epsilon$ in $\mathbf{R}P^2\backslash\mathcal{D}_\epsilon$. Then, there exists ϵ_0 such that for any $0 < \epsilon \leq \epsilon_0$ and any $D(a_i, \epsilon)$ of $\mathcal{D}_\epsilon$ there exists a homeomorphism $h : D(a_i, \epsilon) \to D^1 \times D^1$ (where D^1 is a one-disc of $\mathbf{R}$) such that

$$h(\mathbf{R}\mathcal{H}_{m+1} \cap D(a_i, \epsilon)) = \{(x, y) \in D^1 \times D^1 | x.y = \frac{1}{2}\}$$

(2) Moreover, for any $0 < \epsilon \leq \epsilon_0$, $\mathbf{R}\mathcal{H}_{m+1}\backslash\mathcal{D}_\epsilon$ is a section of the tubular fibration
$N \to \mathbf{R}\mathcal{A}_{m+1}\backslash\mathcal{D}_\epsilon$.

Method

Assume that there exists a singular polynomial $\tilde{X}_{m+1;\tau}$ with good truncation properties and denote by S the set of its singular points. On this assumption, the Harnack curve $\mathcal{H}_{m+1}$ is the image of a smooth section of the tubular fibration $N \to \{(x_0 : x_1 : x_2) \in \mathbf{C}P^2 | \tilde{X}_{m+1;\tau} = 0\}\backslash S$.

Moreover, outside $x_0.B_m = 0$ and outside $C_{m+1} = 0$, curves $\tilde{X}_{m+1;\tau} = 0$ and $\tilde{X}_{m+1;t} = 0$, $0 < t < t_{m+1}$, are isotopic. Thus, $\mathcal{H}_{m+1}$ and $\tilde{X}_{m+1;\tau} = 0$ may be not isotopic only in ϵ-neighborhood $U(p)$ of $\mathcal{H}_{m+1}$ defined from points of faces $\Gamma \subset l_m \subset T_{m+1}$ (see definition 1.8 of Chapter 0).

Denote $U(p_i)$, $1 \leq i \leq m$, the ϵ-neighborhood defined from points of a face $\Gamma_i = \{x \geq (m-i), y \leq i, x + y = m\}$, $\Gamma_i \subset l_m \subset T_{m+1}$. The union $\mathcal{B} = \cup_{i=1}^m U(p_i)$ is a neighborhood of the set of singular points of $\mathbf{C}\mathcal{A}_{m+1}$ in $\mathbf{C}P^2$. Let N be the ϵ-tubular neighborhood of $\mathbf{C}\mathcal{A}_{m+1}\backslash\mathcal{B}$ in $\mathbf{C}P^2\backslash\mathcal{B}$. It follows immediately from the Lemma 3.3 of Chapter 2 that given $U \subset \mathbf{C}P^2$ compact which contains no singular points of $\mathbf{C}\mathcal{A}_{m+1}$, any polynomial $x_0.\tilde{X}_{m,t}$ with $t \in]0, t_m[$, is ϵ-sufficient for $\tilde{X}_{m+1,t}$ in U. In other words, for $t \in]0, t_{m+1}[$, the intersection $U \cap \mathbf{C}\mathcal{H}_{m+1}$ is contained in N and can be extended to the image of a smooth section of a tubular fibration $N \to \mathbf{C}\mathcal{A}_{m+1}\backslash\mathcal{B}$.

According to the corollary 2.4 of Chapter 1 of chapter 1 and its proof, in the patchworking scheme crossings of $\mathcal{H}_m \cup L$ are in bijective correspondence with faces $\Gamma_i = \{x \geq (m-i), y \leq i, x + y = m\}$ of the triangulation of T_{m+1}. Hence, we may assume that any crossing $a_i \in \mathcal{H}_m \cup L$ belongs to the ϵ-neighborhood $U(p_i)$ of $\mathcal{H}_{m+1}$ defined from points Γ_i^0, and consider the truncation of the affine restriction of a Harnack polynomial b_{m+1} on the monomials $x^{m-i}y^{i-1}, x^{m-i}y^i, x^{m-i+1}y^{i-1}, x^{m-i+1}y^i$ which is ϵ-sufficient for b_{m+1} in $U(p_i)$.

Using these truncations, and the fact that for any neighborhood $B(p)$ of a singular point p of $\tilde{X}_{m+1;\tau} = 0$, $\mathbf{C}\mathcal{H}_{m+1} \cap B(p)$ is the image of a smooth section of the tubular fibration
$N \cap B(p) \to \{(x_0 : x_1 : x_2) \in \mathbf{C}P^2 | \tilde{X}_{m+1;\tau} = 0\} \cap B(p)\backslash p$ we shall characterize singular points of curves $\tilde{X}_{m+1;\tau} = 0$. Then, according to the definition of the

vector $\vec{a}_{m+1}$, we shall construct singular polynomials $\tilde{X}_{m+1;\tau}$ with singular points on $\mathcal{B}$ and thus characterize maximal simple deformation of T-Harnack curves.

The exposition of the main ideas of our proof is now finished and we shall now proceed to precise arguments.

In order to construct singular polynomials $\tilde{X}_{m;t}$, $t \in [t_m, t_{m-1}[$ with good truncation properties, we shall distinguish the cases of even and odd m.

We shall work with notations and definitions related to the patchworking theory given in the chapter 1.

i) Consider the case $m = 2k$.

i.1) Let us prove the following Lemma.

LEMMA 3.4. *Any polynomial $\tilde{X}_{2k;\vec{a}_{2k},t}$ which satisfies good truncation properties has at most $2k-3$ singular points. These points are crossings.*

proof:

The proof is based on the Patchworking theory. From a local study on the e-neighborhoods $U(p_i)$ of $\mathcal{H}_{2k}$ defined from points Γ_i^0 of faces $\Gamma_i = \{x \geq (m-i), y \leq i, x+y = m\}$, we shall deduce where singular points of a polynomial $\tilde{X}_{2k;t}$ may appear.

i.a) Let S be a square defined by vertices $(c,d), (c+1,d), (c,d+1), (c+1,d+1)$ with c, d odd and $c+d = 2k-2$. It is contained in the interior of the triangle T_{2k}. (Obviously, there are $(k-1)$ such squares contained in the interior of T_{2k}.)

Consider polynomials $\tilde{X}_{2k;t}$, $t \in]0, t_{2k-1}[$ $\tilde{X}_{2k;t} = x_0.\tilde{X}_{2k-1;\vec{a}_{2k-1},t} + C_{2k;\vec{c}_{2k},t}$ satisfying good truncation properties. (Given the vector $\vec{a}_{2k-1}$, the vector $\vec{c}_{2k} \in ((\mathbf{R}^*)^+)^{2k+1}$ is the vector which may be chosen.)

Denote $x_{2k;t}$ the affine restriction of a polynomial $\tilde{X}_{2k;t}$, $t \in]0, t_{2k-1}[$.

Denote $x^S_{2k;t}(x,y)$ its truncation on the monomials $x^c y^d, x^c y^{d+1}, x^{c+1} y^d, x^{c+1} y^{d+1}$. Namely,

$$x^S_{2k;t}(x,y) = a_{c,d} t^{\nu(c,d)} x^c y^d + a_{c,d+1} t^{\nu(c,d+1)} x^c y^{d+1} + a_{c+1,d} t^{\nu(c+1,d)} x^{c+1} y^d - a_{c+1,d+1} t^{\nu(c+1,d+1)} x^{c+1} y^{d+1}$$

with $a_{c,d} > 0, a_{c+1,d} > 0, a_{c,d+1} > 0, a_{c+1,d+1} > 0$.

Let Γ be the face in the triangulation of T_{2k} given by coordinates $(c, d+1), (c+1, d)$. According to the patchworking construction, it follows that for $t \in]0, t_{2k}[$ the truncation $x^S_{2k;t}$ is ϵ-sufficient for $x_{2k;t}$ in an ϵ-neighborhood $U(p)$ of $x^S_{2k;t} = 0$ defined from points of Γ^0.)

Fix $t \in]0, t_{2k-1}[$, and let a_t be the crossing of $\mathcal{H}_{2k-1} \cup L$ contained in $U(p)$.

According to the patchworking construction, there exists an homeomorphism $\tilde{h} : \mathbf{C}\mathcal{H}_{2k} \cap U(p) \to \{(x,y) \in (\mathbf{C}^*)^2 | x^S_{2k;t} = 0\} \cap U(p)$ such that $\tilde{h}(a_t) = (x_1, y_1)_t$ is a critical point of $x^S_{2k;t}(x,y)$. Obviously, (see corollary 2.4 of Chapter 1), $(x_1, y_1)_t$ is such that $x_1.y_1 < 0$.

Set $x^S_{2k;t}(x,y) = l_t(x,y) + t.k_t(x,y)$ with

$$l_t(x,y) = a_{c,d} t^{\nu(c,d)} x^c y^d + a_{c+1,d} t^{\nu(c,d+1)} x^{c+1} y^d$$
$$k_t(x,y) = a_{c,d+1} t^{\nu(c+1,d)-1} x^c y^{d+1} + -a_{c+1,d+1} t^{\nu(c+1,d+1)-1} x^{c+1} y^{d+1}.$$

Note that up to modify the coefficients $a_{c,d}, a_{c,d+1}, a_{c+1,d}, a_{c+1,d+1}$ if necessary, the point $\tilde{h}(a_t) = (x_1, y_1)_t$ is a critical point of the function $\frac{l_t(x,y)}{k_t(x,y)}$ with positive critical value $\leq t < t_{2k-1}$.

On this assumption, it follows from the equalities
$l_t(x,-y) = -l_t(x,y)$, $k_t(x,-y) = k_t(x,y)$,
$\frac{\partial l_t}{\partial x}(x,-y) = -\frac{\partial l_t}{\partial x}(x,y)$, $\frac{\partial l_t}{\partial y}(x,-y) = \frac{\partial l_t}{\partial y}(x,y)$,
$\frac{\partial k_t}{\partial x}(x,-y) = \frac{\partial k_t}{\partial x}(x,y)$, $\frac{\partial k_t}{\partial y}(x,-y) = -\frac{\partial k_t}{\partial y}(x,y)$
that $(x_1, -y_1)_t$ is a critical point of the function $\frac{l_t(x,y)}{k_t(x,y)}$ with negative critical value $\geq -t > -t_{2k-1}$.

Therefore, for fixed $t \in [t_{2k}, t_{2k-1}[$, one can choose $C_{2k;t}$ and thus $\tilde{X}_{2k;t}$, in such a way that $\tilde{h}^{-1}(x_1, -y_1)_t$ with $x_1.y_1 < 0$ is a singular point of $\tilde{X}_{2k;t} = 0$, and one-parameter curves $\tilde{X}_{2k;t}$ are Harnack curves for $t > 0$ sufficiently small.

Moreover, the singular situation, we shall denote by $\mathcal{S}'$, is as follows:

$\mathcal{S}'$: one outer oval of the part $j_1(\mathbf{R}\mathcal{H}_{2k-1}\backslash\mathcal{D}_\epsilon)$ of $\mathbf{R}\mathcal{H}_{2k}$ touches the non-empty outer oval in the part $\mathbf{R}\mathcal{H}_{2k}\backslash j_1(\mathbf{R}\mathcal{H}_{2k}\backslash\mathcal{D}_\epsilon)$.
(Namely, one branch of a outer oval contained in the patchworking scheme in the subset $\rho^{2k}(D_{2k-1,2k-2}\times U^2_{\mathbf{R}}) \subset \mathbf{R}P^2$ touches a branch of the non-empty outer oval.)

Thus, according to the Proposition 1.4 of Chapter 2 and Petrovskii's theory (Petrovskii's Lemmas (Lemma.2 and Lemma.3)), it follows that the point $\tilde{h}^{-1}(x_1, -y_1)_t$ is a critical point of a Harnack polynomial B_{2k} of type $\mathcal{H}^0$ and belongs to S'_{2k}.

i.b) Likewise, consider squares S defined by vertices
$(c+1,d), (c,d), (c,d+1), (c+1,d+1)$ with $c+d = 2k-2$ and
$c > 0, d > 0$ even. (Obviously, there are $(k-2)$ such squares contained into T_{2k}.) Denote $x^S_{2k;t}(x,y)$ the truncation of a polynomial $x_{2k;t}$ on the monomials $x^c y^d, x^c y^{d+1}, x^{c+1} y^d, x^{c+1} y^{d+1}$. Let Γ be the face in the triangulation of T_{2k} given by coordinates $(c, d+1), (c+1, d)$.

(From the patchworking construction, it follows that for $t \in]0, t_{2k}[$ the truncation $x^S_{2k;t}$ is ϵ-sufficient for $x_{2k;t}$ in an ϵ-neighborhood $U(p)$ of $x^S_{2k;t} = 0$ defined from points of Γ^0.)

Fix $t \in]0, t_{2k-1}[$, and let a_t be a crossing of $\mathcal{H}_{2k-1} \cup L$ contained in $U(p)$.

According to the patchworking construction, there exists an homeomorphism $\tilde{h} : \mathbf{C}\mathcal{H}_{2k} \cap U(p) \to \{(x,y) \in (\mathbf{C}^*)^2 | x^S_{2k;t} = 0\} \cap U(p)$ such that $\tilde{h}(a_t) = (x_1, y_1)_t$ is a critical point of $x^S_{2k;t}(x,y)$. Obviously, (see corollary 2.4 of Chapter 1), $(x_1, y_1)_t$ is such that $x_1.y_1 < 0$.

From an argumentation similar to the one above, keeping the same notations, it follows that, up to modify the coefficients $a_{c,d}, a_{c,d+1}, a_{c+1,d}, a_{c+1,d+1}$ if necessary, one can get a critical point $\tilde{h}(a_t)$ $(x_1, y_1)_t$, $x_1.y_1 < 0$ of the function $\frac{l_t(x,y)}{k_t(x,y)}$ with positive critical value $\leq t < t_{2k-1}$. Therefore, for fixed $t \in]0, t_{2k-1}[$, one can choose $C_{2k;t}$ and thus $\tilde{X}_{2k;t}$, in such a way that $\tilde{h}^{-1}(x_1, -y_1)_t$ with $x_1.y_1 < 0$ is a singular point of $\tilde{X}_{2k;t} = 0$, and one-parameter curves $\tilde{X}_{2k;t}$ are Harnack curves for t sufficiently small.

Moreover, the singular situation, we shall denote by $\mathcal{S}$", is as follows:

$\mathcal{S}$": one branch of a inner oval of the part $j_1(\mathbf{R}\mathcal{H}_{2k-1}\backslash\mathcal{D}_\epsilon)$ of $\mathbf{R}\mathcal{H}_{2k}$ not isotopic to a part of $\mathbf{R}\mathcal{H}_{2k-2}$
(namely, in the patchworking scheme, one inner oval contained in the subset $\rho^{2k}(D_{2k-1,2k-2}\times U^2_{\mathbf{R}})\subset \mathbf{R}P^2$) touches a branch of the non-empty outer oval.

Thus, according to the Proposition 1.4 of Chapter 2 and the Petrovskii's theory, it follows that the point $\tilde{h}^{-1}((x_1,-y_1)_t)$ is a critical point of positive critical value of a Harnack polynomial B_{2k}. (Obviously, $\tilde{h}^{-1}((x_1,-y_1)_t\in S^+_{2k}\backslash S'_{2k}$.)

From an argumentation similar to the previous one, it is easy to deduce that there are no other points which may be singular points of a polynomial $\tilde{X}_{2k;t}$. Otherwise, it would contradict the strict positiveness of the coefficients of the vector $\vec{a}_{2k}$. (Indeed, consider the squares S given by vertices $(c+1,d),(c,d),(c,d+1),(c+1,d+1)$ with $c+d=2k-2$ and $c=0$ or $d=0$. Keeping the previous notations, from a crossing a_t of $\mathcal{H}_{2k-1}\cup L$, one can get a critical point $\tilde{h}(a_t)=(x_1,y_1)_t$ of the function $\frac{l_t(x,y)}{k_t(x,y)}$ with negative critical value.)

Hence, we have obtained the Lemma 3.4 of Chapter 2.

i.2)
Let us now construct a polynomial $\tilde{X}_{2k;\tau}$, $\tau\in[t_{2k},t_{2k-1}[$ with $2k-3$ singular points: $k-1$ points described locally by the singular situation $\mathcal{S}'$, and $k-2$ points described locally by the singular situation $\mathcal{S}$".

Let us fix $\tau<t_{2k-1}$. Crossings of the union $\mathcal{H}_{2k-1}\cup L$ belong to the line L.

Let $p_i=\tilde{h}^{-1}(x,-y)_\tau$ with $(x,y)_\tau$ defined up to homeomorphism from a crossing of the curve $\mathcal{H}_{2k-1}\cup L$ with polynomial $x_0.\tilde{X}_{2k-1;\vec{a}_{2k-1},\tau}$. We shall prove that given the $2k-3$ points p_1,...,p_{2k-3} the polynomial $\tilde{X}_{2k;\tau}$ may be entirely defined.

According to the previous description of points $p_1,...,p_{2k-3}$, we shall assume that the $2k-3$ singular points $\tilde{X}_{2k,\tau}$ belong to the line $x'_0=0$ with $x'_0=x_0+\alpha.x_1+\beta.x_2$. Then, consider the linear change of complex projective coordinates mapping $(x_0:x_1:x_2)$ to $(x'_0:x_1:x_2)$. Such transformation carries $x_0.\tilde{X}_{2k-1,\tau}(x_0,x_1,x_2)+C_{2k,\tau}$ to $\tilde{X}'_{2k,\tau}=x'_0.\tilde{X}'_{2k-1,\tau}(x'_0,x_1,x_2)+C'_{2k,\tau}$. We shall prove that one can choose $C'_{2k}(x_1,x_2)=0$ a union of m parallel lines in such a way that $C'_{2k}(x_1,x_2)=0$, $\tilde{X}'_{2k-1}(x'_0,x_1,x_2)=0$, and $x'_0=0$ have $2k-3$ common points.

In such a way, using the linear change of projective coordinates mapping $(x'_0:x_1:x_2)$ to $(x_0:x_1:x_2)$, we shall get the polynomial $\tilde{X}_{2k,\tau}(x_0,x_1,x_2)=x_0.\tilde{X}_{2k-1,\tau}(x_0,x_1,x_2)+C_{2k}(x_1,x_2)$ from the polynomial $\tilde{X}'_{2k,\tau}(x_0,x_1,x_2)=x'_0.\tilde{X}'_{2k-1,\tau}(x'_0,x_1,x_2)+C'_{2k}(x_1,x_2)$ and in that way the vector $\vec{c}_{2k}$.

Let us detail this construction.

Assume singular points $p=(0:p_1:p_2)$ of $\tilde{X}'_{2k}=0$ such that $\frac{\partial\tilde{X}'_{2k}}{\partial x'_0}(p)=0$ (i.e belong to the curve $C'_{2k}=0$.)
Assume $(\frac{\partial\tilde{X}'_{2k-1}}{\partial_{x'_0}}=0)\cap(x'_0=0)\cap(\tilde{X}'_{2k-1}=0)=\emptyset$,

Set

$$D_{2k}(x)=x_1.\frac{\partial\tilde{X}'_{2k}}{\partial x_1}+x_2.\frac{\partial\tilde{X}'_{2k}}{\partial x_2}$$

$$D_{2k-1}(x)=x_1.\frac{\partial^2\tilde{X}'_{2k}}{\partial x'_0\partial x_1}+x_2.\frac{\partial^2\tilde{X}'_{2k}}{\partial x'_0\partial x_2}$$

where $D_{2k-1}(0,x_1,x_2)$ is an homogeneous polynomial of degree $(2k-1)$ in the variables x_1,x_2.

Since $(\frac{\partial \tilde{X'}_{2k-1}}{\partial_{x'_0}}=0)\cap(x'_0=0)\cap(\tilde{X'}_{2k-1}=0)=\emptyset$, $\frac{\partial^2\tilde{X'}_{2k}}{\partial^2 x'_0}(p)\neq 0$ for any singular point p of $\tilde{X'}_{2k}$.

Thus, for $2k>2$, $D_{2k}(x'_0,x_1,x_2)$ is of degree at least 1 in x'_0 and any point p belongs to $\frac{\partial D_{2k}}{\partial_{x'_0}}=D_{2k-1}=0$.

According to the Euler formula, for any singular point p of $\tilde{X}'_{2k}$ the following equalities are verified:

$$D_{2k}(p)=p_1.\frac{\partial\tilde{X}'_{2k}}{\partial x_1}(p)+p_2.\frac{\partial\tilde{X}'_{2k}}{\partial x_2}(p)=0 \tag{11}$$

$$D_{2k-1}(p)=p_1.\frac{\partial^2\tilde{X}'_{2k}}{\partial x'_0\partial x_1}(p)+p_2.\frac{\partial^2\tilde{X}'_{2k}}{\partial x'_0\partial x_2}(p)=0 \tag{12}$$

$$\frac{\partial\tilde{X}'_{2k}}{\partial x_1}(p)=0 \quad \textit{or} \quad \frac{\partial\tilde{X}'_{2k}}{\partial x_2}(p)=0 \tag{13}$$

It is easy to deduce from the Newton's binomial formula (applied to $(x'_0)^i=(x_0+\alpha.x_1+\beta.x_2)^i$. that coefficients of the polynomial $\tilde{X}'_{2k}=x'_0.\tilde{X}'_{2k-1}+C'_{2k}$ which depend on the vector $\vec{c}_{2k}$ (and do not depend on $\vec{c}_{2k-1}$) are coefficients of monomials $x_1^{2k-i}x_2^i$, (namely coefficients of C'_{2k}) and coefficients of monomials $x'_0.x_1^{2k-1-i}x_2^i$ and in that way coefficients of $D_{2k-1}(0,x_1,x_2)$.

Therefore, any singular point of $\tilde{X}'_{2k}=0$ belongs to the intersection of $x'_0=0$ with $C'_{2k}=0$ and $\tilde{X}'_{2k-1}=0$. Let us detail the construction of $\tilde{X}'_{2k}$.

Since any singular point p of $\tilde{X}'_{2k}$ belongs to $O_1=\{(0:x_1:x_2)\in\mathbf{C}P^2|x_1\neq 0\}$ or $O_2=\{(0:x_1:x_2)\in\mathbf{C}P^2|x_2\neq 0\}$, it is sufficient to consider singular points of $\tilde{X}'_{2k}$ in the affine charts associated to O_1 and O_2.

It follows from the form of $\tilde{X}'_{2k}(x_0,x_1,x_2)$ that points $(0:0:1)$ and $(0:1:0)$ are not singular points of the polynomial $\tilde{X}'_{2k}(x_0,x_1,x_2)$. Consequently, any singular point p of $\tilde{X}'_{2k}$ is such that

$$\frac{\partial\tilde{X}'_{2k}}{\partial x_1}(p)=0$$

or

$$\frac{\partial\tilde{X}'_{2k}}{\partial x_2}(p)=0$$

Up to constant factor, the following equalities are verified:

$$\frac{\partial\tilde{X}'_{2k}}{\partial x_2}(0,x_1,x_2)=x_2^{2k-1}-c_1x_2^{2k-2}x_1+...(-1)^{2k-1}c_{2k-1}x_1^{2k-1} \tag{14}$$

where c_i, $i \in \{1, ..., 2k-1\}$ is the elementary roots symmetric polynomial of degree i of $\frac{\partial \tilde{X}'_{2k}}{\partial x_2}(0, 1, x_2)$

$$\frac{\partial \tilde{X}'_{2k}}{\partial x_1}(0, x_1, x_2) = x_1^{2k-1} - s_1 x_1^{2k-2} x_2 + ...(-1)^{2k-1} s_{2k-1} x_2^{2k-1} \tag{15}$$

where s_i, $i \in \{1, ..., 2k-1\}$ is the elementary roots symmetric polynomial of degree i of $\frac{\partial \tilde{X}'_{2k}}{\partial x_1}(0, x_1, 1)$.

Set $x_1 = 1$ (respectively, $x_2 = 1$) in the equality (14) (respectively, (15)). Bringing together the resulting equalities, it follows that $\tilde{X}'_{2k}(0, x_1, 1)$ has at most $[\frac{2k-1}{2}]$ singular points (Indeed, any singular point $(0, p_1, 1)$ is of multiplicity 1 of $D_{2k-1}(0, x_1, 1)$ and of multiplicity at least 2 of $\tilde{X}'_{2k,\tau}(0, x_1, 1)$). Besides, $\tilde{X}'_{2k}(0 : 1 : x_2)$ has at most $[\frac{2k-1}{2}]$ singular points (since any singular point $(0, 1, p_2)$ is of multiplicity 1 of $D_{2k-1}(0, 1, x_2)$ and of multiplicity at least 2 of $\tilde{X}'_{2k,\tau}(0, 1, x_2)$)).

Since $\tilde{X}'_{2k} = 0$ is singular, it follows immediately from the equalities (14), (15) and (12), (13) that at least one point $p = (0 : 1 : -1) = (0 : -1 : 1)$ is a singular point of $\tilde{X}'_{2k}$.

Therefore, the construction of the polynomial $\tilde{X}'_{2k;\tau}(x_0, x_1, x_2)$ with $2k-3$ singular points p_1,...,p_{2k-3} on the intersection of $x'_0 = 0$ with $C'_{2k} = 0$ and $\tilde{X}'_{2k-1} = 0$ follows immediately.

Moreover, according to the Lemma 3.3 of Chapter 2, one can perturb coefficients of one monomials x_1^{2k} or x_2^{2k} of the polynomial $\tilde{X}'_{2k}$ in such a way that the modified polynomial is a Harnack polynomial of degree $2k$ with critical points p_1,..,p_{2k-3}.

This concludes the part (2) of proposition 3.1 of Chapter 2.

ii) Consider the case $m = 2k+1$.

From an argumentation similar to the previous one, it is easy to deduce that any polynomial $\tilde{X}_{2k+1;t}$, $t \in]0, t_{2k}[$ which satisfies good truncation properties is a Harnack polynomial. Besides, it is also obvious that the existence of a singular polynomial $\tilde{X}_{2k+1;t}(x_0, x_1, x_2).x_0 + C_{2k+1;t}(x_1, x_2)$ with good truncation properties is not compatible with the Harnack's distribution of signs (i.e the strict positiveness of the coefficients of the vector $\vec{a}_{2k+1}$.)

Indeed, consider the squares S given by vertices
$(c+1, d), (c, d), (c, d+1), (c+1, d+1)$ with $c+d = 2k-1$, $c \geq 0$ and $d \geq 0$. Keeping the notations introduced in the case $m = 2k$, from a crossing a_t of $\mathcal{H}_{2k} \cup L$, one can get a critical point $\tilde{h}(a_t) = (x_1, y_1)_t$ of the function $\frac{l_t(x,y)}{k_t(x,y)}$ with negative critical value. Therefore, from an argument similar to the one given in case $m = 2k$, it follows easily that any polynomial $\tilde{X}_{2k+1;t}(x_0, x_1, x_2).x_0 + C_{2k+1;t}(x_1, x_2)$, $t \in]0, t_{2k}[$, with good truncation properties is smooth.
Q.E.D

Maximal Deformation of any Harnack curves. We shall now in Proposition 3.6 of Chapter 2 describe maximal simple deformation of any Harnack curve.
Let us start with Corollary 3.5 of Chapter 2 of Theorem 2.1 of Chapter 2.

COROLLARY 3.5. *Let $\mathcal{H}_m$ be a Harnack curve of degree m. Then, one can assume, without changing its topological properties, that any Harnack curve $\mathcal{H}_m$ is obtained via the T-inductive construction of Harnack curves.*

proof: It is straightforward consequence of the rigid isotopy Theorem 2.1 of Chapter 2. Q.E.D.

The generalization of Proposition 3.1 of Chapter 2 to any Harnack curve is a straightforward consequence of the Corollary 3.5 of Chapter 2.

PROPOSITION 3.6. *Denote $\mathcal{H}_m$ the Harnack curve of degree m defined up to isotopy of real points set.*

(1) *Along any simple deformation of a Harnack curve of odd degree $2k+1$ curves are smooth.*
(2) *Given $\mathcal{H}_{2k}$ a Harnack curve of even degree $2k$, any maximal simple deformation of $\mathcal{H}_{2k}$*

$$[0,1] \to \mathbf{C}\mathcal{C}_m$$
$$t \to \mathcal{H}_{m;t}$$

is such that $\mathcal{H}_{2k;1}$ has $2k-3$ crossings.
Let S be the set of singular points of the singular curve $\mathcal{H}_{2k;1}$. Then, there exists a Harnack polynomial B_{2k} of degree $2k$ and type $\mathcal{H}^0$ of which the set of critical points contains the set S.
These points are critical points with positive critical value. Besides, $k-1$ of these points are points of S'_{2k}.

proof: The generalization of Proposition 3.1 of Chapter 2 is a straightforward consequence of the Corollary 3.5 of Chapter 2.
We propose here a generalization of the argumentation given in the proof of proposition 3.6 of Chapter 2.

Consider $B_m(x_0,x_1,x_2)$ a Harnack polynomial of degree m.

According to Proposition 2.31 of Chapter 2, it is possible to choose projective coordinates such that $B_m(x_0,x_1,x_2) = x_0.B_{m-1}(x_0,x_1,x_2) + C_m(x_1,x_2)$ where $B_{m-1}(x_0,x_1,x_2)$ is the Harnack polynomial of degree $m-1$.

According to Morse Lemma [**15**], around any non-degenerate singular point s of $B_{m-1}.x_0$ one can choose a local coordinates system y_1, y_2 with $z_1(s)=0, z_2(s)=0$ and $x_0.B_{m-1}(x_0,x_1,x_2) = y_1.y_2$.

Analogously, in a neighborhood of any critical point p of B_m, one can choose a local coordinates system y_1, y_2 with $y_1(p)=0, y_2(p)=0$ and $B_m(x_0,x_1,x_2) = B_m(p)+y_1.y_2$. Therefore, given a singular crossing s of $x_0.B_{m-1}$, since s is critical point of the curve $B_m = 0$ considered as level curve of the function $\frac{x_0.B_{m-1}}{C_m}$ the crossing s is perturbed in such a way that around s, one can choose local coordinates in an open $U(s)$ with $y_1(s)\neq 0, y_2(s)\neq 0$ and $B_m(x_0,x_1,x_2) = \frac{y_1}{y_2}+t$.
Such description enlarges to a description of $B_m(x_0,x_1,x_2) = 0$ inside $U \supset U(s)$ $U = \{z =< u,p >= (u_0.p_0 : u_1.p_1 : u_2.p_2) \in \mathbf{C}P^2 | u = (u_0 : u_1 : u_2) \in U^3_{\mathbf{C}}, p = (p_0 : p_1 : p_2) \in U(s)\}$ Local coordinates y_1, y_2 inside the open $U(s)$ extend to local coordinates inside U as follows. Given $z =< u,p >\in U$ with $u = (1 : u_1 : u_2) \in U^3_{\mathbf{C}}$, $p = (p_0 : p_1 : p_2) \in U(s)\}$, we may set $y_1(z) = u_1.y_1(p)$, $y_2(z) = u_2.y_2(p)$.

In such a way, given a singular crossing s of $x_0.B_{m-1}$, any point $p \in U$ with local coordinates $y_1(p), y_2(p)$ such that $|y_1(p)| = |y_1(s)|$, $|y_2(p)| = |y_2(s)|$ and $y_1(p).y_2(p) = -y_1(s).y_2(s)$, is a critical point of $B_m(x_0, x_1, x_2)$.

Moreover, according to Rolle's Theorem, around any singular point s of $x_0.B_{m-1}$, in a neighborhood of the line $x_0 = 0$, the sign of $B_m(0, x_1/x_2, 1)$ alternates. (One can refer also to the marking method (see [**10**]) where a distribution of sign of C_m and $x_0.B_{m-1}$ is defined around any point of intersection of C_m with $x_0.B_{m-1}$.) Therefore, when consider the critical points of B_m deduced locally from singular points of $x_0.B_{m-1}$, the sign of $B_m(0, x_1/x_2, 1)$ alternates as well around these points. By continuity of B_m, it follows the alternation of sign of $B_m(x_0, x_1, x_2)$ in an ϵ-tubular neighborhood of the line $x_0 = 0$ in $\mathbf{C}P^2$. It is obvious that such alternation of sign is equivalent to the modular property of the distribution of sign in the patchworking construction of Harnack's curves. Hence, our proposition is a straightforward consequence of the version of proposition 3.6 of Chapter 2 for T-Harnack curves. For sake of clarity, we refer to proposition 3.1 of Chapter 2 of this chapter where a proof of the version of proposition 3.6 of Chapter 2 for T-Harnack curves is given.

Hence, it follows from proposition 3.1 of Chapter 2, that along any simple deformation of a Harnack curve of odd degree $2k+1$ curves are smooth. From an argumentation similar to the one given in the second part of proof of it follows that in case $m = 2k$ one can modify coefficients of the polynomial in such a way that it has at most $2k-3$ singular points.

Q.E.D

Description of $(\mathbf{C}P^2, \mathbf{C}\mathcal{H}_m)$ up to conj-equivariant isotopy. Let us recall in introduction some properties of the real point set of Harnack curves which may be easily deduced from the Harnack curves construction and, according to proposition 3.6 of Chapter 2, may be generalized to any Harnack curve

Given $\mathcal{A}_{m+1} = \mathcal{H}_m \cup L$. Let $\mathcal{D}_\epsilon = \cup_{a_i \in \mathcal{H}_m \cap L} D(a_i, \epsilon) \in \mathbf{R}P^2$ be a neighborhood of the set of singular points of $\mathcal{A}_{m+1}$ in $\mathbf{R}P^2$ (where $D(a_i, \epsilon)$ denotes a 2-disc of radius ϵ around a_i in the Fubini-Study metric).

Let N be a tubular neighborhood of $\mathbf{R}\mathcal{A}_{m+1} \backslash \mathcal{D}_\epsilon$ in $\mathbf{R}P^2 \backslash \mathcal{D}_\epsilon$. Then, the Harnack curve $\mathcal{H}_{m+1}$ is a non-singular curve of degree $m+1$ such that: $\mathbf{R}\mathcal{H}_{m+1} \backslash \mathcal{D}_\epsilon$ is a section of the tubular fibration $N \to \mathbf{R}\mathcal{A}_{m+1} \backslash \mathcal{D}_\epsilon$.

Therefore, there exists $\epsilon > 0$ and $\tilde{j}_t$, with $t \in [0,1]$, an isotopy of $\mathbf{R}P^2$ which pushes $\mathbf{R}\mathcal{H}_m \backslash \mathcal{D}_{\epsilon_0}$ onto a subset of $\mathbf{R}\mathcal{H}_{m+1}$.

Besides, there is the following biunivoque correspondence between critical points of index 0 and 2 of Harnack polynomials B_m and ovals of curves:

(1) ovals of the part $\tilde{j}_1(\mathbf{R}\mathcal{H}_m \backslash \mathcal{D}_\epsilon)$ of $\mathbf{R}\mathcal{H}_{m+1}$ with ovals of $\mathcal{H}_m$ and thus with critical points of index 0 and 2 of B_{m+1}.
(2) m ovals of $\mathbf{R}\mathcal{H}_{m+1} \backslash \tilde{j}_1(\mathbf{R}\mathcal{H}_m \backslash \mathcal{D}_\epsilon)$ with critical points of index 0 and 2 of B_{m+1}.

Any oval of these m ovals intersects a 2-disc $D(a_i) \subset \mathbf{R}P^2$ defined around a point $a_i \in \mathcal{H}_m \cap L$.

We shall in Proposition 3.7 of Chapter 2 define the minimal (minimal in the sense of the inclusion of subsets of $\mathbf{C}P^2$), subset $\mathcal{B}_\epsilon$ of $\mathbf{C}P^2$ which consists of union

of 4-ball (in the Fubini-Study metric) of $\mathbf{C}P^2$ of radius ϵ, $\mathcal{D}_\epsilon \subset \mathcal{B}_\epsilon$, outside of which the isotopy $\tilde{j}_t$, $t \in [0,1]$, extends to a conj-equivariant isotopy $\mathbf{C}P^2$ which pushes $\mathbf{C}\mathcal{H}_m \backslash \mathcal{B}$ onto a subset of $\mathbf{C}\mathcal{H}_{m+1}$.

In such a way, we shall enlarge the above description of Harnack curves to the complex domain and then in theorem 3.9 of Chapter 2 describe the pair $(\mathbf{C}\mathcal{H}_m, \mathbf{C}P^2)$ up to conj-equivariant isotopy of $\mathbf{C}P^2$.

In way of preparation, let us make some remarks and recall generalities connected with Harnack curves and complex topological characteristics of real curves.

A curve $\mathcal{A}_m$ of type I is such its real set of points $\mathbf{R}\mathcal{A}_m$ divides its complex set of points into two connected pieces called halves $\mathbf{C}\mathcal{A}_m^+$, $\mathbf{C}\mathcal{A}_m^-$. These two halves are exchanged by complex conjugation $conj(\mathbf{C}\mathcal{A}_m^+) = \mathbf{C}\mathcal{A}_m^-$. The natural orientations of these two halves determine two opposite orientations on $\mathbf{R}\mathcal{A}_m$ (as their common boundary), called the *complex orientations* of $\mathcal{A}_m$.
It is easy to see that the deformation turning $\mathcal{H}_m \cup L$, into $\mathcal{H}_{m+1}$ brings the complex orientations of $\mathcal{H}_m$ and L to the orientations of the corresponding pieces of $\mathbf{R}\mathcal{H}_{m+1}$ induced by a single orientation of the whole $\mathbf{R}\mathcal{H}_{m+1}$.
In such a way, given an orientation of $\mathbf{R}\mathcal{H}_m$, there exists only one orientation of $\mathbf{R}L$ which induces the orientation of $\mathbf{R}\mathcal{H}_{m+1}$. Moreover, it is easy to deduce from the recursive construction of Harnack curves that orientations of $\mathbf{R}L$ alternate. Namely, given an orientation of $\mathbf{R}\mathcal{H}_{m-1}$, $m > 1$, if $\mathbf{C}L^+$ is the half of $\mathbf{C}L$ which induces the orientation on $\mathbf{R}L$ carried on $\mathbf{R}\mathcal{H}_m$, then $\mathbf{C}L^- = conj(\mathbf{C}L^+)$ induces the orientation on $\mathbf{R}L$ carried on $\mathbf{R}\mathcal{H}_{m+1}$.

Let us recall the different ways to perturb one crossing p of a singular curve $\mathcal{A}$. In a neighborhood invariant by complex conjugation of the crossing p , the curve $\mathcal{A}$ can be considered up to conj-equivariant isotopy as the intersection of two lines in the point p. Thus, let L_1 and L_2 be lines embedded in $\mathbf{C}P^2$ which intersect each other in a single point p. Denote by C the result of the perturbation of the union $L_1 \cup L_2$. From the complex viewpoint, there are essentially two ways to perturb the singular union $L_1 \cup L_2$. On the other hand, there are two ways to connect the halves of their complexifications. Indeed, the curve C is a curve of type I since it is a non-empty conic. The halves of $\mathbf{C}L_i$ $i \in \{1,2\}$ connected each other after perturbation correspond to the complex orientation of $\mathbf{R}L_i$ which agrees with some perturbation of $\mathbf{R}C$. One distinguishes two ways to connect the halves with each other. We shall define them as *perturbation of type 1 and type 2* of the crossing (see figure 1.1 and figure 1.2). Let B be a complex 4-ball globally invariant by complex conjugation around the crossing. After a *perturbation of type* 1, $\mathbf{R}C \cap B$ does not divide $\mathbf{C}C \cap B$ into two connected pieces. After a perturbation of type 2, $\mathbf{R}C \cap B$ divides $\mathbf{C}C \cap B$ into two connected pieces.

According to Theorem 2.30 of Chapter 2, one can assume that Harnack polynomials are deduced by induction on the degree as follows:
i) The Harnack polynomial B_1 is the polynomial of a real projective line.
Given B_{2k-1} a Harnack polynomial of degree $2k-1 \geq 3$. Then, (up to linear change of projective coordinates and up to slightly modify its coefficients), B_{2k-1} is a Harnack polynomial of degree $2k-1$, and type $\mathcal{H}^0$ deduced from classical small deformation $B_{2k-1} = x_0.B_{2k-2} + \epsilon_{2k}C_{2k-1}$ of $x_0.B_{2k-2}$ where B_{2k-2} is Harnack polynomial of degree $2k-2$ and type $\mathcal{H}^0$. We shall denote $\mathcal{A}_{2k-1} = \mathcal{H}_{2k-2} \cup L$ the

curve given by $x_0.B_{2k-2}$.
For $k \geq 1$, we shall denote A_{2k-1} and call *set of points perturbed in a maximal simple deformation of* $\mathcal{H}_{2k-1}$ the set of points $A_{2k-1} = \{a_1, ..., a_{2k}\}$ where $a_1, ...a_{2k}$ are the crossings of $\mathcal{H}_{2k-2} \cup L$.

ii) The Harnack polynomial B_2 is the polynomial of a curve of degree 2 of which real part consists of an oval. We shall denote A_2 the set $A_2 = \{a_1\}$ where a_1 is the crossing of $\mathcal{H}_1 \cup L$.

Given B_{2k} a Harnack polynomial of degree $2k \leq 4$. Then, (up to linear change of projective coordinates and up to slightly modify its coefficients), B_{2k} is a Harnack polynomial of degree $2k$ and type $\mathcal{H}^0$ deduced from classical small deformation $B_{2k} = x_0.B_{2k-1} + C_{2k}$ (with $||C_{2k}||$ arbitrarily small) of $B_{2k-1}.x_0$, where B_{2k-1} is a Harnack polynomial of degree $2k-1$ and type $\mathcal{H}^0$. We shall denote $\mathcal{A}_{2k} = \mathcal{H}_{2k-1} \cup L$ the curve given by $x_0.B_{2k-1}$.

Let $\mathcal{H}_{2k;t}$, $t \in [0,1]$, be a maximal simple deformation of the Harnack curve $\mathcal{H}_{2k}$ given by the polynomial B_{2k}. For $k \geq 2$, we shall denote A_{2k} and call *set of points perturbed in a maximal simple deformation of* $\mathcal{H}_{2k}$ the ordered set of points $A_{2k} = \{a_1, ..., a_{2k-1}, ..., a_{4k-2}\}$ where $a_1, ...a_{2k-1}$ are the crossings of $\mathcal{H}_{2k-1} \cup L$ and $a_{2k}, ..., a_{4k-2}$ are the crossings of $\mathcal{H}_{2k;1}$.

One can easily notice that any point of A_m is perturbed by a perturbation of type 1 to give $\mathcal{H}_m$. (i.e locally around any crossing of $\mathcal{H}_m \cup L$, the deformation turning $\mathcal{H}_m \cup L$ into $\mathcal{H}_{m+1}$ is a perturbation of type 1 of the crossing. In case of even $m = 2k$, given a maximal simple deformation $\mathcal{H}_{2k,t}$, $0 \leq t \leq 1$, of $\mathcal{H}_{2k}$; locally around any crossing a of $\mathcal{H}_{2k;1}$, the deformation turning $\mathcal{H}_{2k;1} = 0$ into $\mathcal{H}_{2k} = 0$ is a perturbation of type 1 of a.)

Let $\mathbf{C}\mathcal{H}_m^+$ (resp, $\mathbf{C}L^+$) be the half of $\mathbf{C}\mathcal{H}_m$ (resp, $\mathbf{C}L$) which induces orientation on $\mathbf{R}\mathcal{H}_m$ (resp,$\mathbf{R}L$). Denote $\mathbf{C}\mathcal{A}_m^+$ the union $\mathbf{C}\mathcal{H}_m^+ \cup \mathbf{C}L^+$.

In proposition 3.7 of Chapter 2, we construct a subset $\mathcal{B}_\epsilon$ of $\mathbf{C}P^2$, (minimal in the sense of the inclusion of subsets of $\mathbf{C}P^2$), $\mathcal{D}_\epsilon \subset \mathcal{B}_\epsilon$, with the property that outside $\mathcal{B}_\epsilon$, the isotopy $\tilde{j}_t$ which pushes $\mathbf{R}\mathcal{H}_m \backslash \mathcal{D}_\epsilon$ onto a subset of $\mathbf{R}\mathcal{H}_{m+1}$ extends to a a conj-equivariant isotopy of $\mathbf{C}P^2$ which maps $\mathbf{C}\mathcal{H}_{m+1}^+ \backslash \mathcal{B}_\epsilon$ to $\mathbf{C}\mathcal{H}_m^+ \cup \mathbf{C}L^+ \backslash \mathcal{B}_\epsilon$. The construction of $\mathcal{B}_\epsilon$ is based on the definition of a maximal simple deformation of $\mathcal{H}_m$ and the way points of the set A_m are perturbed to give $\mathcal{H}_m$.

PROPOSITION 3.7. (1) *Let $\mathcal{B}_\epsilon = \cup_{a \in A_{2k+1}} B_(a, \epsilon) \subset \mathbf{C}P^2$, the union of conj-equivarant 4-ball $B_(a, \epsilon)$ of radius ϵ taken over crossings a of $\mathcal{A}_{2k+1} = \mathcal{H}_{2k} \cup L$, be a neighborhood of the set of singular points of $\mathcal{A}_{2k+1} = \mathcal{H}_{2k} \cup L$ in $\mathbf{C}P^2$. Let N be an oriented tubular neighborhood of $\mathbf{C}\mathcal{A}_{2k+1}^+ \backslash \mathcal{B}_\epsilon$ in $\mathbf{C}P^2 \backslash \mathcal{B}_\epsilon$.*
Then, there exists ϵ_0 such that, $\mathbf{C}\mathcal{H}_{2k+1}^+ \backslash \mathcal{B}_{\epsilon_0}$ is the image of a non-zero section of the oriented tubular fibration $N \to \mathbf{C}\mathcal{A}_{2k+1}^+ \backslash \mathcal{B}_{\epsilon_0}$ and there exists j_t with $t \in [0,1]$ a conj-equivariant isotopy of $\mathbf{C}P^2$ which maps $\mathbf{C}\mathcal{H}_{2k+1}^+ \backslash \mathcal{B}_{\epsilon_0}$ to $\mathbf{C}\mathcal{H}_{2k}^+ \cup \mathbf{C}L^+ \backslash \mathcal{B}_{\epsilon_0}$.

(2) *Let* $\mathcal{B}_\epsilon = \cup_{a\in A_{2k}} B(a,\epsilon) \subset \mathbf{C}P^2$ *the union of conj-equivarant 4-ball* $B(a,\epsilon)$ *of radius* ϵ *taken over points* a *of* A_{2k}*,be a neighborhood in* $\mathbf{C}P^2$ *of the set of points perturbed in a maximal simple deformation of* $\mathcal{H}_{2k}$*. (*$\mathcal{B}_\epsilon$ *contains besides neighborhoods of singular point of* $\mathcal{A}_{2k} = \mathcal{H}_{2k-1} \cup L$*, the union of neighborhood of* $2k-3$ *crossings which appear in a maximal simple deformation of* $\mathcal{H}_{2k}$*.) Let* N *be an oriented tubular neighborhood of* $\mathbf{C}\mathcal{A}^+_{2k}\backslash\mathcal{B}_\epsilon$ *in* $\mathbf{C}P^2\backslash\mathcal{B}_\epsilon$*.*
Then, there exists ϵ_0 *such that* $\mathbf{C}\mathcal{H}^+_{2k}\backslash\mathcal{B}_{\epsilon_0}$ *is the image of a non-zero section of the oriented tubular fibration* $N \to \mathbf{C}\mathcal{A}^+_{2k}\backslash\mathcal{B}_{\epsilon_0}$ *and there exists* j_t*, with* $t \in [0,1]$*, a conj-equivariant isotopy of* $\mathbf{C}P^2$ *which maps* $\mathbf{C}\mathcal{H}^+_{2k}\backslash\mathcal{B}_{\epsilon_0}$ *to* $\mathbf{C}\mathcal{H}^+_{2k-1} \cup \mathbf{C}L^+\backslash\mathcal{B}_{\epsilon_0}$*.*

proof:

Our proof is based on results of proposition 3.1 of Chapter 2 and the fact that orientation of $\mathbf{R}L$ alternates in the recursive construction of Harnack curves.

Let us explain briefly the method of our proof.

We shall consider the usual handlebody decomposition of $\mathbf{C}P^2 = B_0 \cup B_1 \cup B$ where B_0,B_1,B are respectively 0,2 and 4 handles. In such a way, the canonical $\mathbf{R}P^2$ can be seen as the union of a Möbius band $\mathcal{M}$ and the disc $D^2 \subset B$ glued along their boundary. The Möbius band $\mathcal{M}$ lies in $B_0 \cap B_1 \approx S^1 \times D^2$. The complex conjugation switches B_0 and B_1 and $\mathcal{M}$ belongs to the set of its fixed points.

Let $\mathcal{D}_\epsilon$ be a neighborhood of the set of singular points of $\mathcal{A}_{m+1}$ in $\mathbf{C}P^2$ which is the union of 2-disc of radius ϵ around point of the set $\mathcal{A}_m$.

We shall assume that there exists an isotopy $\tilde{j}_t$ of $\mathbf{R}P^2$ which maps $\mathbf{R}\mathcal{H}_{m+1}\backslash\mathcal{D}$ on the boundary of the Möbius band and study whether or not $\tilde{j}_t$ extends to j_t a conj-equivariant isotopy of $\mathbf{C}P^2$ which maps $\mathbf{C}\mathcal{H}^+_{m+1}\backslash\mathcal{D}_\epsilon$ to $\mathbf{C}\mathcal{H}^+_m \cup \mathbf{C}L^+\backslash\mathcal{D}_\epsilon$.

We shall notice that given a 4-ball $B(p) \subset (B_0 \cup B_1) \subset \mathbf{C}P^2$ globally invariant by complex conjugation and such that $B(p)^- = B(p) \cap B_1$ and $B(p)^+ = B(p) \cap B_0$, if $\mathbf{C}\mathcal{H}^+_{m+1} \cap B(p) \neq \emptyset$ and $\mathbf{C}\mathcal{H}^+_{m+1} \cap B(p) \neq \mathbf{C}\mathcal{H}^+_{m+1} \cap B(p)^\pm$ then there is no conj-equivariant isotopy j_t of $\mathbf{C}P^2$ which maps $\mathbf{C}\mathcal{H}^+_{m+1}\backslash\mathcal{D}_\epsilon \cap B(p)$ to $((\mathbf{C}\mathcal{H}^+_m \cup \mathbf{C}L^+)\backslash\mathcal{D}_\epsilon) \cap B(p)$.

Thereby, since orientation of $\mathbf{R}L$ alternates in the recursive construction of Harnack curves an obstruction to construct a conj-equivariant isotopy in a 4-ball $B(p)$ will be provided by a crossing $a \in B(p)$ of a curve $\mathcal{H}_{m+1;1}$ where $\mathcal{H}_{m+1;t}$ $t \in [0,1]$ is a simple deformation of the Harnack curve $\mathcal{H}_{m+1}$.

We shall now proceed to precise arguments.
Assume $m+1 = 2k \geq 4$.

First, we shall prove that the existence of a conj-equivariant isotopy j_t which maps $\mathbf{C}\mathcal{H}^+_{2k}\backslash\mathcal{D}_\epsilon$ to $\mathbf{C}\mathcal{H}^+_{2k-1} \cup \mathbf{C}L^+\backslash\mathcal{D}_\epsilon$ is not compatible with elementary topological properties of $\mathbf{C}\mathcal{H}_{2k}$. Then, using proposition 3.1 of Chapter 2, we shall define a subset $\mathcal{B}_\epsilon \subset \mathbf{C}P^2$, $\mathcal{B}_\epsilon \supset \mathcal{D}_\epsilon$, such that there exists a conj-equivariant isotopy j_t which maps $\mathbf{C}\mathcal{H}^+_{2k}\backslash\mathcal{B}_\epsilon$ to $\mathbf{C}\mathcal{H}^+_{2k-1} \cup \mathbf{C}L^+\backslash\mathcal{B}_\epsilon$.

Since $\mathcal{H}_{2k}$ is of even degree, its real point set $\mathbf{R}\mathcal{H}_{2k}$ divides the projective plane $\mathbf{R}P^2$ in two components respectively orientable and non-orientable whose common boundary is $\mathbf{R}\mathcal{H}_{2k}$.

Without loss of generality, one can assume that $D^2 \subset \mathbf{R}P^2$ is a small real disc with boundary the line at infinity L of $\mathbf{R}P^2$ with $\mathcal{D}_\epsilon \cap \mathbf{R}P^2 \subset D^2$. The tubular neighborhood of the line L is the Möbius band $\mathcal{M}$ of $\mathbf{R}P^2$. Consider $\tilde{j}_t$, with $t \in [0,1]$ an isotopy of $\mathbf{R}P^2$ which lets fix the interior of the disc D^2 of $\mathbf{R}P^2$ and pushes the rest of $\mathbf{R}\mathcal{H}_{2k}\backslash\mathcal{D}_\epsilon$ to the boundary of the Möbius band.
Assume that the isotopy $\tilde{j}_t$ extends to a conj-equivariant isotopy j_t of $\mathbf{C}P^2$. Then, one can set $j_1(\mathbf{C}\mathcal{H}_{2k}^+\backslash B) \subset B_0$. Given N a tubular neighborhood of $j_1(\mathbf{C}\mathcal{H}_{2k})$ in $\mathbf{C}P^2$, $j_1(\mathbf{C}\mathcal{H}_{2k})$ is the image of a non-zero section of the oriented tubular fibration $N \to j_1(\mathbf{C}\mathcal{H}_{2k})$.

Since j_t is conj-equivariant, there exists $N^+ \subset B_0$ a tubular neighborhood of $\mathbf{C}\mathcal{H}_{2k}^+\backslash B$ in $\mathbf{C}P^2\backslash\mathcal{D}_\epsilon$ such that
$N^+ \to \mathbf{C}\mathcal{H}_{2k}^+\backslash B$ is an oriented tubular fibration.

Hence, since $\mathcal{H}_{2k}$ is two-sided and has orientable real part, there exists N an oriented tubular neighborhood of $\tilde{j}_1(\mathbf{R}\mathcal{H}_{2k}\backslash\mathcal{D}_\epsilon)$ in $\mathbf{R}P^2\backslash\mathcal{D}_\epsilon$ such that $\tilde{j}_1(\mathbf{R}\mathcal{H}_{2k}\backslash\mathcal{D}_\epsilon$ is the image of a non-zero smooth section of the tubular fibration $N \to \tilde{j}_1(\mathbf{R}\mathcal{H}_{2k}\backslash\mathcal{D}_\epsilon)$.

It leads to contradiction since the tubular neighborhood of $j_1(\mathbf{R}\mathcal{H}_{2k}\backslash(D^2)^0)$ in $\mathbf{R}P^2\backslash(D^2)^0$ is homeomorphic to the Möbius band and therefore has non-orientable normal fibration.

Therefore, on $\mathbf{R}\mathcal{H}_{2k}\backslash\mathcal{D}_\epsilon$, the isotopy $\tilde{j}_t$ does not extend to a conj-equivariant isotopy j_t of $\mathbf{C}P^2$.

Nonetheless, using the proposition 3.6 of Chapter 2, we shall define a set $\mathcal{B}_\epsilon \supset \mathcal{D}_\epsilon$ for which on $\mathbf{R}\mathcal{H}_{2k}\backslash\mathcal{B}$ the isotopy $\tilde{j}_t$ extends to a conj-equivariant isotopy j_t which maps $\mathbf{C}\mathcal{H}_{2k}^+\backslash\mathcal{B}_\epsilon$ to $\mathbf{C}\mathcal{H}_{2k-1}^+ \cup \mathbf{C}L^+\backslash\mathcal{B}_\epsilon$.

Given $\mathcal{H}_{2k}$ the Harnack curve of degree $2k$, consider $\mathcal{H}_{2k;t}$, $t \in [0,1]$, a simple deformation of the Harnack curve $\mathcal{H}_{2k}$. Let us prove the following Lemma.

LEMMA 3.8. *There exists $B(p) \subset B_0 \cup B_1$ such that $\mathbf{C}\mathcal{H}_{2k}^+ \cap B(p) \neq \mathbf{C}\mathcal{H}_{2k}^+ \cap B(p)^\pm$ if and only if there exists a simple deformation $\mathcal{H}_{2k;t}$ $t \in [0,1]$ of the Harnack curve $\mathcal{H}_{2k}$ such that a crossing of $\mathcal{H}_{2k;1}$ belongs to $B(p)$.*

proof:

For $\epsilon > 0$ small, the sets of complex points of curves $\mathcal{H}_{2k;t}$ $t \in [1-\epsilon, 1]$ belong to a tubular neighborhood of the set of complex points of the Harnack curves $\mathcal{H}_{2k;1}$.

Let $B(p) \subset \mathbf{C}P^2$ be a 4-ball centered in a singular point of $\mathcal{H}_{2k;1}$ such that $\mathbf{C}\mathcal{H}_{2k;t\ ,t\in[1-\epsilon,1]} \cap B(p) \neq \emptyset$. Inside $B(p)$, consider the gradient trajectories of the deformation turning $\mathcal{H}_{2k;1}$ into $\mathcal{H}_{2k;1-\epsilon}$. Up to isotopy $\tilde{j}_t$ of $\mathbf{R}P^2$, one can assume $B(p) \subset B_0 \cup B_1$ and set $B(p)^- = B(p) \cap B_1$ and $B(p)^+ = B(p) \cap B_0$.

Since orientation of the line L alternates in the recursive construction of Harnack curves, the complex conjugation acts on gradient trajectories as symmetry of center p. Therefore, inside $B(p)$ we have $\mathbf{C}\mathcal{H}_{2k}^+ \cap B(p) \neq \mathbf{C}\mathcal{H}_{2k}^+ \cap B(p)^\pm$. (i.e around $B(p)$, the deformation turning $\mathcal{H}_{2k;1}$ into $\mathcal{H}_{2k;1-\epsilon}$ is a deformation of type 1.)

Reciprocally, it is easy to see that if there exists $B(p) \subset B_0 \cup B_1$ such that $\mathbf{C}\mathcal{H}_{2k}^+ \cap B(p) \neq \mathbf{C}\mathcal{H}_{2k}^+ \cap B(p)^\pm$ then there exists a simple deformation $\mathcal{H}_{2k;t}$ $t \in [0,1]$

such that a crossing of $\mathcal{H}_{2k;1}$ belongs to $B(p)$.
Q.E.D

The Lemma 3.8 of Chapter 2 implies that outside a neighborhood $\mathcal{B}_\epsilon$ of the set of singular points of $\mathcal{H}_{2k;1}$, $\tilde{j}_t$ with $t \in [0,1]$ extends to a conj-equivariant isotopy of $\mathbf{C}P^2$ which maps $\mathbf{C}\mathcal{H}^+_{2k} \backslash \mathcal{B}_\epsilon$ to $\mathbf{C}\mathcal{H}^+_{2k-1} \cup \mathbf{C}L^+ \backslash \mathcal{B}_\epsilon$.

Assume now $m+1 = 2k+1 \geq 3$,

Without loss of generality one can assume that D^2 is a small real disc such that $D^2 \cap \mathcal{H}_{2k+1} \cup L = \mathcal{D}$. Consider $\tilde{j}_t$, with $t \in [0,1]$, an isotopy of $\mathbf{R}P^2$ which lets fix the disc D^2 of $\mathbf{R}P^2$ and pushes $\mathbf{R}\mathcal{H}_{2k+1} \backslash \mathcal{D}_\epsilon$ to the boundary of the Möbius band. As any curve of odd degree, $\mathcal{H}_{2k+1}$ is one-sided. Therefore, we may not use the argument given in case of curve of even degree to refute the existence of a conj-equivariant isotopy j_t of $\mathbf{C}P^2$ which maps $\mathbf{C}\mathcal{H}^+_{2k+1} \backslash \mathcal{D}_\epsilon$ to $\mathbf{C}\mathcal{H}^+_{2k} \cup \mathbf{C}L^+ \backslash \mathcal{D}_\epsilon$.

Besides, from an argument analogous to the one given in even degree, using the odd version of the previous Lemma 3.8 of Chapter 2, it follows that on $\mathbf{R}\mathcal{H}_{2k} \backslash \mathcal{D}_\epsilon$, $\tilde{j}_t$ extends to a conj-equivariant isotopy j_t of $\mathbf{C}P^2$.

Otherwise, there would exist a simple deformation $\mathcal{H}_{2k+1;t}$ $t \in [0,1]$ of the Harnack curve $\mathcal{H}_{2k+1}$ of degree $2k+1$ which intersects the discriminant hypersurface $\mathbf{R}\mathcal{D}_{2k+1}$. According to proposition 3.6 of Chapter 2, it is impossible. Q.E.D

The following theorem is a straighforward consequence of the Proposition 3.7 of Chapter 2.

THEOREM 3.9. *Let $\mathcal{H}_m$ be the Harnack curve of degree m. There exists a finite number I ($I = 1 + 2...m + \Sigma_{k=2}^{k=[m/2]} 2k-3$) of disjoint 4-balls $B(a_i)$ invariant by complex conjugation centered in points a_i of $\mathbf{R}P^2$ such that up to conj-equivariant isotopy of $\mathbf{C}P^2$,*

(1) *$\mathcal{H}_m \backslash \cup_{i \in I} B(a_i) = \cup_{i=1}^m L_i \backslash \cup_{i=1}^I B(a_i)$ where $L_1, ..., L_m$ are m distinct projective lines with*

$$L_i \backslash \cup_{i=1}^I B(a_i) \cap L_j \backslash \cup_{i=1}^I B(a_i) = \emptyset$$

for any $i \neq j$, $1 \leq i,j \leq m$.

(2) *situation inside any 4-ball $B(a_i)$ is described by the perturbation of type 1 of the crossing a_i.*

proof:

For $m=1$, the theorem is trivially verified: $\mathcal{H}_1$ is a projective line. For $m>1$, it may be deduced by induction on m. The induction is based on the proposition 3.7 of Chapter 2 and the inductive construction of Harnack curves.

As related above, for each integer m ($m \geq 0$), on can assume without loss of generality, that the curve $\mathcal{H}_{m+1}$ results from classical deformation of $\mathcal{H}_m \cup L$.

According to the proposition 3.6 of Chapter 2,

(1) let $A_{2k} = \{a_1, ..., a_{2k-1}, ..., a_{4k-2}\}$ be the set of points perturbed in a maximal simple deformation of $\mathcal{H}_{2k}$
(2) denote $A_{2k+1} = \{a_1, ..., a_{2k}\}$ the set of points perturbed in a maximal simple deformation of $\mathcal{H}_{2k+1}$

For any $a_i \in A_{m+1}$, we may always choose a small open conj-symmetric 4-ball $B(a_i)$ of a_i, in such a way that $B(a_i) \subset \rho^{m+1}(D(p_j) \times U^2_{\mathbf{C}})$ with $D(p_j)$ is a small open real disc around $p_j \in \Gamma_j$.

(1) Given $\mathcal{H}_{2k-1}$ the Harnack curve of degree $2k-1$, from the proposition 3.7 of Chapter 2, it results the following characterization of $\mathcal{H}_{2k}$:

There exists j_t, $t \in [0,1]$, a conj-equivariant isotopy of $\mathbf{C}P^2$, which maps $\mathcal{H}_{2k} \setminus \cup_{i=1}^{4k-2} B(a_i)$ onto

(a) $\mathcal{H}_{2k-1} \setminus \cup_{i=1}^{4k-2} B(a_i)$ (in the patchworking scheme, such part is contained in the restriction $\rho^{2k}(T_{2k-1} \times U^2_{\mathbf{C}})$ of $\mathbf{C}P^2 \approx \mathbf{C}T_{2k}$.)

(b) union $L \setminus \cup_{i=1}^{4k-2} B(a_i)$ (in the patchworking scheme, such part is contained in the restriction $\rho^{2k}(D_{2k,2k-1} \times U^2_{\mathbf{C}})$ of $\mathbf{C}P^2 \approx \mathbf{C}T_{2k}$.)

(2) Given $\mathcal{H}_{2k}$ the Harnack curve of degree $2k$. From the proposition 3.7 of Chapter 2, it results the following characterization of $\mathcal{H}_{2k+1}$:

There exists j_t, $t \in [0,1]$, a conj-equivariant isotopy of $\mathbf{C}P^2$, which maps $\mathcal{H}_{2k+1} \setminus \cup_{i=1}^{2k} B(a_i)$ onto

(a) $\mathcal{H}_{2k} \setminus \cup_{i=1}^{2k} B(a_i)$ (in the patchworking scheme, such part is contained in the restriction $\rho^{2k+1}(T_{2k} \times U^2_{\mathbf{C}})$ of $\mathbf{C}P^2 \approx \mathbf{C}T_{2k+1}$.)

(b) union $L \setminus \cup_{i=1}^{2k} B(a_i)$ (in the patchworking scheme, such part is contained in the restriction $\rho^{2k+1}(D_{2k+1,2k} \times U^2_{\mathbf{C}})$ of $\mathbf{C}P^2 \approx \mathbf{C}T_{2k+1}$.)

Therefore, it follows easily by induction on m that outside a finite number of 4-balls $B(a_i)$ the curve $\mathcal{H}_m$ is, up to conj-equivariant isotopy, the union of m (non-intersecting) projective lines minus their intersections with 4-balls $B(a_i)$.

The 4-balls $B(a_i)$ are centered in points a_i. Inside any 4-ball $B(a_i)$, one can get the whole description of $\mathcal{H}_{m+1}$ from the real set of points $\mathbf{R}\mathcal{H}_{m+1} \cap B(a_i)$ and its orientation (i.e the type of deformation).

Situations inside 4-balls centered in crossings a_i of $\mathcal{H}_m \cup L$ are easily deduced from the construction of $\mathcal{H}_{m+1}$: the deformation turning $(\mathcal{H}_{2k} \cup L) \cap B(a_i)$ into $\mathcal{H}_{2k+1} \cap B(a_i)$ is of type 1.

In case $m+1 = 2k$, one has to consider 4-balls centered in crossings $B_{2k;1}$. The situation inside 4-balls centered in crossings $B_{2k;1}$ has been explicitly described inside $D(a_i)$, and respectively inside $B(a_i)$, in the proof of proposition 3.1 of Chapter 2, and respectively in the proof of proposition 3.7 of Chapter 2. The deformation turning $\mathcal{H}_{2k;1} \cap B(a_i)$ into $\mathcal{H}_{2k} \cap B(a_i)$ is of type 1.

Q.E.D

CHAPTER 3

Arnold surfaces of Harnack curves

In this chapter, we valid in Theorem 0.10 of Chapter 3 the Rokhlin Conjecture for the Harnack curves of even degree. In other words, we prove in Theorem 0.10 of Chapter 3 that Arnold surfaces of Harnack curves split into connected sum of $\mathbf{R}P^2$ and $\overline{\mathbf{R}P^2}$. This result is a straightforward consequence of Theorem 3.9 of Chapter 2.

THEOREM 0.10. *Arnold surfaces of Harnack curves of even degree are standard. Moreover, let $\mathcal{H}_{2k}$ be the Harnack curve of degree $2k$ and $\mathfrak{A}_+$, $\mathfrak{A}_-$ its Arnold surfaces then, up to isotopy of S^4: for $k = 1$*

$$\mathfrak{A}_+ \approx D^2$$
$$\mathfrak{A}_- \approx \overline{\mathbf{R}P^2}$$

for $k \geq 2$

$$\mathfrak{A}_+ \approx p\mathbf{R}P^2 \# q_+ \overline{\mathbf{R}P^2}$$
$$\mathfrak{A}_- \approx p\mathbf{R}P^2 \# q_- \overline{\mathbf{R}P^2}$$

where
$p = q_+ = \frac{(k-1)(k-2)}{2}; q_- = \frac{k(5k-3)}{2}$

proof:
The proof, is based on the theorem 3.9 of Chapter 2 stated previously, and on the next Livingston's statement:

Theorem [**14**] *: Let $F \subset D^4 \subset S^4$ be a closed surface which lies in $S^3 = \partial D^4$, except several two-dimensional discs standardly embedded inside D^4. Then F is a standard surface in S^4.*

Let us recall in details the handles decomposition of $\mathbf{C}P^2$ and $\mathbf{C}P^2/conj \approx S^4$.

Choose b a point in $\mathbf{R}P^2$ and B a small neighborhood of b in $\mathbf{C}P^2$. Setting $b = (b_0 : b_1 : 1) \in \mathbf{R}P^2$, B is defined, up to homeomorphism, in $U_2 = \{(x_0 : x_1 : x_2) \in \mathbf{C}P^2 | x_2 \neq 0\} \subset \mathbf{C}P^2$ $U_2 \approx \mathbf{C}^2$ as a usual 4-ball of $\mathbf{C}^2$: $B \approx \{p = (p_0 : p_1 : 1) \in \mathbf{C}^2 \mid ||(p_0 - b_0, p_1 - b_1)|| \leq \epsilon\}$.
The 4-ball B is globally invariant by complex conjugation. The real set of points of the ball B is a two-dimensional disc D^2 which remains fix by complex conjugation.

Then, consider the central projection

$$p : \mathbf{C}P^2 \backslash B \to \mathbf{C}L$$

from b to some set of complex points of a real projective line L, $b \notin \mathbf{C}L$. Denote $\mathbf{C}L_0$ and $\mathbf{C}L_1$ the two connected parts of $\mathbf{C}L$ such that :

(1) $\mathbf{C}L = \mathbf{C}L_0 \cup \mathbf{C}L_1$
(2) $\mathbf{C}L_0 \cap \mathbf{C}L_1 = \mathbf{R}L$

Since $\mathbf{C}L_0 \approx D^2$,$\mathbf{C}L_1 \approx D^2$, $\mathbf{R}L \approx S^1$, the fibrations:

$$p^{-1}(\mathbf{C}L_0) \to \mathbf{C}L_0 \approx D^2$$
$$p^{-1}(\mathbf{C}L_1) \to \mathbf{C}L_1 \approx D^2$$
$$p^{-1}(\mathbf{R}L) \to \mathbf{R}L \approx S^1$$

are, up to homeomorphism, trivial fibrations.

Therefore, setting $p^{-1}(\mathbf{C}L_0) = B_0$ and $p^{-1}(\mathbf{C}L_1) = B_1$ it follows :

$$B_0 \approx D^2 \times D^2, B_1 \approx D^2 \times D^2, B_0 \cap B_1 = p^{-1}(\mathbf{R}L) \approx S^1 \times D^2$$

Thus, we shall take the handlebody decomposition of $\mathbf{C}P^2$ $\mathbf{C}P^2 = B_0 \cup B_1 \cup B$ where B_0,B_1,B are respectively 0,2 and 4 handles.

Moreover, from the action of complex conjugation on $\mathbf{C}P^2 = B_0 \cup B_1 \cup B$ it is easy to deduce a decomposition of $\mathbf{C}P^2/conj \approx S^4$.

The canonical $\mathbf{R}P^2$ can be seen as the union of a Möbius band $\mathcal{M}$ and the disc D^2 glued along their boundary. In such a way, $\mathcal{M}$ lies in $B_0 \cap B_1 \approx S^1 \times D^2$.

The complex conjugation switches B_0 and B_1 and $\mathcal{M}$ belongs to the set of its fixed points. The quotient $(B_0 \cup B_1)/conj$ is a 4-ball as well which contains the quotient $(B_0 \cap B_1)/conj$.

Consider now $\mathcal{H}_{2k}$ the Harnack curve of degree $2k$ and $\mathfrak{A}_+$ and $\mathfrak{A}_-$ its Arnold surfaces.

Acording to Theorem 3.9 of Chapter 2, there exists a conj-equivariant isotopy of $\mathbf{C}P^2$, h_t, $t \in [0,1]$, $h(0) = \mathcal{H}_{2k}$ and a finite number I of disjoint 4-balls $B(a_i)$ invariant by complex conjugation centered in points a_i of $\mathbf{R}P^2$ such that:

(1) $h_1(\mathcal{H}_{2k} \backslash \cup_{i=1}^{I} B(a_i)) = \cup_{i=1}^{2k} L_i \backslash \cup_{i=1}^{I} B(a_i)$ where $L_1, ..., L_{2k}$ are $2k$ distinct projective lines with
$$(L_i \backslash \cup_{i=1}^{I} B(a_i)) \cap (L_j \backslash \cup_{i=1}^{I} B(a_i)) = \emptyset$$
for any i, j, $1 \leq i \neq j \leq 2k$,
(2) inside any 4-ball $B(a_i)$, $h_1(\mathcal{H}_m \cap B(a_i))$ is the perturbation of a crossing as described in theorem 3.9 of Chapter 2.

Choose the 4-ball B of the handlebody decomposition of $\mathbf{C}P^2$ centered in b in such a way that:

(1) $b \in \mathbf{R}P^2_{\pm}$ (For example, choose $b = (0, 0, 1)$ inside $\mathbf{R}P^2_-$ and $b = (\epsilon, \epsilon, 1)$, $0 < \epsilon < 1$, inside $\mathbf{R}P^2_+$).
(2) B does not intersect with $\cup_{i \in I} B(a_i)/conj$. and lines $L_1, ..., L_{2k}$ provided by the isotopy h_t.

Then, delete a small tubular neighborhood U of $(B_0 \cap B_1)/conj$ from $(B_0 \cup B_1)/conj$ and denote by B' the resulting 4-ball.

From the decomposition of $\mathbf{C}P^2/conj \approx S^4$ and the conj-equivariant isotopy h_t, it follows an isotopy $h_t/conj$ of S^4 which pushes

$\mathbf{R}P^2_{\pm}\backslash \cup_{i=1}^{I} B(a_i)/conj$ on the boundary of B' and $(\mathcal{H}_{2k}/conj\backslash \cup_{i=1}^{I} B(a_i)/conj)$ to $2k$ standards discs of B'. (such discs are defined by quotient of lines $L_1/conj$, ...,$L_{2k}/conj$ minus their intersection with $\cup_{i\in I} B(a_i)/conj$.)

Thus, since $\mathfrak{A}_{\pm}\backslash \cup_{i\in I} B(a_i)/conj \cap IntU = \emptyset$, the problem of construction of an isotopy which pushes $\mathfrak{A}_{\pm}$ to the boundary of a 4-disc and to several 2-discs standardly embedded is reduced to the problem of construction of such an isotopy inside $\cup_{i\in I} B(a_i)/conj$.

Moreover, since the 4-balls $B(a_i)$ are disjoints, it is sufficient to study local questions inside 2-discs $B(a_i)/conj$.

Inside any 4-ball $B(a_i)$, $h_1(\mathcal{H}_{2k} \cap B(a_i))$ is described by the perturbation of the singular crossing a_i.

Thus, inside any 2-disc $B(a_i)/conj$,
$h_1/conj(\mathcal{H}_{2k}/conj \cup \mathbf{R}P^2_{\pm} \cap B(a_i)/conj)$ is determined by the relative positions of the point a_i, the point b and $\mathbf{R}P^2_{\pm}$.

Using an argumentation similar to the one given in [[**8**], see p.6], in any case it is always possible to push by an isotopy the part $\mathfrak{A}_{\pm}$, contained inside $B(a_i)/conj$ in $\partial B(a_i)/conj$ leaving a 2-disc inside $B(a_i)/conj$ if necessary.

Hence, the part $\mathfrak{A}_{\pm}$, contained inside U, can be pushed by an isotopy (which coincides with $h_t/conj$ inside $S^4\backslash \cup_{i=1}^{I} B(a_i)/conj$ and inside any $B(a_i)/conj$ satisfies the requirement above) into a 4-ball B" obtained by taking union of B' with disc $B(a_i)/conj$ or excising $B(a_i)/conj$. (The choice of union or excision of the disc $B(a_i)/conj$ depends on relative position of b and $\mathbf{R}P^2_{\pm} \cap B(a_i)/conj$ [see [**8**]])

After such isotopy, $\mathfrak{A}_{\pm}$ lies in the boundary of B", except $2k$ discs left from $\mathcal{H}_{2k}/conj\backslash \cup_{i=1}^{I} B(a_i)/conj$ and several discs which lie all inside B" and are unknotted.

Thus from Livingston's Theorem it follows that :

Arnold surfaces $\mathfrak{A}_{\pm}$ of Harnack curve $\mathcal{H}_{2k}$ are standard surfaces in S^4.

The decomposition announced in theorem 0.10 of Chapter 3 is immediate when considers the two double coverings branched respectively over $\mathfrak{A}_-$ and $\mathfrak{A}_+$ (see [**2**]), However, we propose to recover this decomposition from the study of local situations inside 4-balls $B(a_i)$ centered in crossings points a_i perturbed in maximal simple deformation of the Harnack curves.

LEMMA 0.11. *Let $\mathcal{C}_0$ be a singular curve of degree m with one singular crossing. Consider a variation $\mathcal{C}_t$, $t \in [-1, 1]$ which crosses transversally the discriminant Δ_m at $\mathcal{C}_0$ Let p be the singular crossing of $\mathcal{C}_0$. Let $\mathfrak{A}_t$ denote either the positive or negative Arnold surface of $\mathcal{C}_t$. Assume that inside a small neighborhood of p:*

(1) *For $t \leq 0$, $\mathfrak{A}_t$ is the union of $\mathcal{C}_t/conj$ and a disc component.*
(2) *For $t > 0$, $\mathfrak{A}_t$ is the union of $\mathcal{C}_t/conj$ and a Möbius component. Then the perturbation $\mathcal{C}_0$ to $\mathcal{C}_1$ corresponds on Arnold surface to $\#\overline{\mathbf{R}P^2}$ (The direction of twisting of the Möbius band is obviously standard.)*

proof:

It can be easily deduced from an algebraic model. In a neighborhood of p, one can choose coordinates such that $\mathcal{C}_t$ is given by $x_1^2 - x_2^2 = t$, the projection $q : \mathbf{C}P^2 \to \mathbf{C}P^2 \backslash conj$ is given by the map $\mathbf{C}^2 \to \mathbf{C}^2$ $(x_1, x_2) \to (x_1^2, x_2)$. Q.E.D

Lemma 0.12. *Let $\mathcal{C}_0$ be a singular curve of degree $2k$ with two crossings p_1 and p_2 such that each p_i $i \in \{1, 2\}$ belongs to a real branch and an oval (with a given orientation);the two real branches which contain p_i have opposite orientation. (see figure 3.1)*

Consider the deformation $\mathcal{C}_t$, $t \in [0, 1]$ which crosses transversally the discriminant $\mathbf{R}\mathcal{D}_m$ at $\mathcal{C}_0$. Then the deformation $\mathcal{C}_0$ to $\mathcal{C}_1$ implies $\mathfrak{A}_1 = \mathfrak{A}_0 \# \mathbf{R}P^2 \# \overline{\mathbf{R}P^2}$.

proof:

In case of Arnold surface $\mathfrak{A}_-$, it may be deduced from the preceding Lemma 0.11 of Chapter 3. It is easy to see that the directions of twisting of the Möbius band given by p_1 and p_2 are opposite.

In case of Arnold surface $\mathfrak{A}_+$, consider the Morse-function $f : \mathcal{C}_0 \times] - 1, +1[\approx \mathcal{C}_{t, t \in]-1,1[} \to] - 1, +1[$, $f(\mathcal{C}_t) = t$.

Obviously, up to isotopy, one can identify $\mathcal{C}_1$ with any curve $\mathcal{C}_{1-\epsilon}$ $\epsilon > 0$ small. Let us fix such $\epsilon > 0$.

Consider the descending one-manifolds D_t of p_1 and D'_t of p_2. Each of them reaches the boundary of the quotient curve $\mathcal{C}_{1-\epsilon} \backslash conj$ and defines a normal bundle N of the real set point $\mathbf{R}\mathcal{C}_{1-\epsilon}$ of $\mathcal{C}_{1-\epsilon}$. Choose a smooth tangent vector field V on $\mathbf{R}P^2$ such that on $\mathbf{R}\mathcal{C}_{1-\epsilon}$ it is to tangent to $\mathbf{R}\mathcal{C}_{1-\epsilon}$ and directed according to the orientation of $\mathbf{R}\mathcal{C}_{1-\epsilon}$. In such a way, for $x \in \mathbf{R}\mathcal{C}_{1-\epsilon}$, $N(x) = iV(x)$ is directed inside the half of $\mathbf{C}\mathcal{C}_{1-\epsilon}$ which induces orientation on $\mathbf{R}\mathcal{C}_{1-\epsilon}$. Recall that multiplication by i makes a real vector normal to the real plane and leaves any vector tangent to $\mathbf{R}\mathcal{C}_{1-\epsilon}$ tangent to $\mathbf{C}\mathcal{C}_{1-\epsilon}$. Extend the tangent vector field V to a tangent vector field of $\mathcal{C}_{1-\epsilon+s}$,$0 < s < \epsilon, s + e < 1$, in a neighborhood of $\mathbf{R}\mathcal{C}_{1-\epsilon} \subset \mathbf{R}P^2_+$ (where $\mathbf{R}P^2_+ = \{x \in \mathbf{R}P^2 | C_{1-\epsilon}(x) \geq 0\}$ with $C_{1-\epsilon}$ is a polynomial giving $\mathcal{C}_{1-\epsilon}$.)

Since complex conjugation acts on gradient-trajectories of D_t (resp D'_t) as symmetry of center p_1 (resp, p_2), the perturbation of $\mathcal{C}_0$ to $\mathcal{C}_{1-\epsilon}$ implies on Arnold surface $\mathfrak{A}_+$ one connected sum with the fibre bundle of the fibration with base space $\mathbf{R}\mathcal{C}_{1-\epsilon}$ and fiber defined for any point $x \in \mathbf{R}\mathcal{C}_{1-\epsilon}$ as the gradient-trajectory through x quotiented by the action of complex conjugation.

Namely, it adds $\# \mathbf{R}P^2 \# \overline{\mathbf{R}P^2}$ on Arnold surface $\mathfrak{A}_+$. Obviously, an argument analogous to this last one may be also used to describe Arnold surfaces $\mathfrak{A}_-$.

Q.E.D

From the Lemmas 0.11 of Chapter 3 and 0.12 of Chapter 3, we shall get topological effect of perturbations of crossings in the recursive construction of Harnack curves on Arnold surfaces and thus the description given in Theorem 0.10 of Chapter 3. We shall use the results of proposition 3.6 of Chapter 2. and (according to the notations introduced in proposition 3.6 of Chapter 2) we shall denote by $S"_{2k}$ the complementary subset of S'_{2k} inside S. It consists of the $(k-2)$ points of $S^+_{2k} \backslash S'_{2k}$.

Let $A_{2k} = \{a_1, ..., a_{2k-1}, ..., a_{4k-2}\}$ be the set of points perturbed in a maximal simple deformation of $\mathcal{H}_{2k}$ (i.e the ordered set of points a_i where $a_1, ...a_{2k-1}$ are the crossings of $\mathcal{H}_{2k-1} \cup L$ and $a_{2k}, ..., a_{4k-2}$ are the crossings of $B_{2k;1}$). Let $A_{2k+1} =$

$\{a_1, ..., a_{2k}\}$ be the set of points perturbed in a maximal simple deformation of $\mathcal{H}_{2k+1}$ (i.e the set of crossings a_i of $\mathcal{H}_{2k} \cup L$).

For any $k \geq 1$, denote $\mathfrak{A}_-^{2k}$ and $\mathfrak{A}_+^{2k}$ the Arnold surfaces. of the Harnack curve $\mathcal{H}_{2k}$ of degree $2k$.

For $k = 1$, it follows easily from the Lemma 0.11 of Chapter 3 that: $\mathfrak{A}_-^2 \approx \overline{\mathbf{R}P^2}$ and $\mathfrak{A}_+^2 \approx D^2$

For $k > 1$, pairs $(S^4, \mathfrak{A}_-^{2k})$, $(S^4, \mathfrak{A}_+^{2k})$ may be deduced by induction on k.

Let $\mathcal{H}_{2k-2}$ be the Harnack curve of even degree $2k - 2 \geq 1$ and $\mathfrak{A}_-^{2k-2}$ and $\mathfrak{A}_+^{2k-2}$ be its Arnold surfaces.

Let $\mathcal{H}_{2k}$ be the Harnack curve of even degree $2k$. and $\mathfrak{A}_-^{2k}$ and $\mathfrak{A}_+^{2k}$ be its Arnold surfaces.

Then, one can deduce the pair $(S^4, \mathfrak{A}_-^{2k})$, (resp,$(S^4, \mathfrak{A}_+^{2k})$) from the pair $(S^4, \mathfrak{A}_-^{2k-2})$ (resp,$(S^4, \mathfrak{A}_+^{2k-2})$).

The Harnack curve $\mathcal{H}_{2k}$ is obtained from $\mathcal{H}_{2k-2}$ by intermediate construction of $\mathcal{H}_{2k-1}$ (see proposition 3.1 of Chapter 2 and theorem 3.9 of Chapter 2 of this chapter).

From Lemma 0.11 of Chapter 3, perturbations of singular crossings of $\mathcal{H}_{2k-2} \cup L$ and then of $\mathcal{H}_{2k-1} \cup L$ imply $(2k-2) + (2k-1) \#\overline{\mathbf{R}P^2}$ on $\mathfrak{A}_-^{2k}$, and imply $(2k-2) + (2k-1) \# D^2$ on $\mathfrak{A}_+^{2k}$.

From the Lemma 0.11 of Chapter 3 and Lemma 0.12 of Chapter 3, keeping the notations of the proposition 3.1 of Chapter 2, it follows that perturbations of singular crossings of $S'_{2k} \subset S^+_{2k}$ imply $(k-1) \#\overline{\mathbf{R}P^2}$ on $\mathfrak{A}_-^{2k}$, and imply $(k-1)$ $\# D^2$ on $\mathfrak{A}_+^{2k}$.

Furthermore, the perturbations of the $(k-2)$ singular crossings of $S"_{2k} \subset S^+_{2k}$ imply
(joined with perturbations of singular crossings of $S'_{2k-2} \subset S^+_{2k-2}$)
$(k-2) \# \mathbf{R}P^2$ on $\mathfrak{A}_-^{2k}$

From the Lemma 0.12 of Chapter 3, it follows that perturbations of singular crossings of
$S"_{2k} \subset S^+_{2k}$ joined with perturbations of singular crossings of $S'_{2k-2} \subset S^+_{2k-2}$ imply $(k-2) \# \mathbf{R}P^2 \# \overline{\mathbf{R}P^2}$ on $\mathfrak{A}_+^{2k}$.

Hence $\mathfrak{A}_-^{2k} \approx \mathfrak{A}_-^{2k-2} \# (k-2) \# \mathbf{R}P^2 \# (5k-4)\overline{\mathbf{R}P^2}$ and $\mathfrak{A}_+^{2k} \approx \mathfrak{A}_+^{2k-2} \# (k-2) \# \mathbf{R}P^2 \# (k-2)\overline{\mathbf{R}P^2}$.

Q.E.D

Part 3

Perestroika Theory on Harnack curves

As already noticed in the last section, the classification problem of pairs $(S^4, \mathcal{A}_\pm)$ amounts to the classification problem of real algebraic curves up to conjugate equivariant isotopy of $\mathbf{C}P^2$.

Thus, there is an obvious connection with classification of Arnold surfaces up to isotopy of S^4 and Hilbert's Sixteen Problem on the arrangements of ovals of real algebraic curves.

First of all we shall detail the construction of Harnack curves. Then, we shall deduce from this detailed construction, a construction of curves $\mathcal{A}_m$ of degree m which provides a description of the pair $(\mathbf{C}P^2, \mathbf{C}\mathcal{A}_m)$ up to conj-equivariant isotopy of $\mathbf{C}P^2$ (see Theorem 2.9 of Chapter 5 and Theorem 3.7 of Chapter 5). Moreover, this method gives all possible arrangements of real connected components of curves and therefore is an advancement in the Hilbert's Sixteen Problem.

After that, we shall deal with Arnold surfaces defined on any curve of even degree with non-empty real part.

CHAPTER 4

Recursive Morse-Petrovskii's theory of recursive Harnack curves

In this section, we shall go on the study of critical points of Harnack's polynomial initiated in chapter 2.
Recall that given a Harnack polynomial $B_m(x_0, x_1, x_2) = x_0^m.b_m(x_1/x_0, x_2/x_0)$ we call critical point of $B_m(x_0, x_1, x_2)$ any point (x_0, y_0) such that $b_x(x_0, y_0) = 0$ and $b_y(x_0, y_0) = 0$. We shall consider only Harnack curves given by regular polynomials. According to the classification of Harnack curves $\mathcal{H}_m$ up to rigid isotopy (Theorem 2.1), one can assume that any Harnack curve $\mathcal{H}_m$ results of the recursive construction of Harnack curves $\mathcal{H}_i$, $1 \leq i \leq m$, where the curve $\mathcal{H}_{i+1}$ is deduced from classical small perturbation of the union $\mathcal{H}_i \cup L$ of the curve $\mathcal{H}_i$ with a line L. This section until its end is devoted to settle how critical points of a Harnack polynomial of degree m may be associated with crossings of curves $\mathcal{H}_{m-i} \cup L$, $1 \leq i \leq (m-1)$, and how a critical point associated with a crossing of $\mathcal{H}_{m-i} \cup L$, $1 \leq i \leq (m-1)$, varies in the recursive construction of Harnack curves $\mathcal{H}_m$.

Given a regular Harnack polynomial $B_m(x_0, x_1, x_2)$ with affine associated polynomial $b_m(x_1/x_0, x_2/x_0)$, we shall study the pencil of curves given by polynomial $x_0^m.(b_m(x_1/x_0, x_2/x_0) - c)$, $c \in \mathbf{R}$. Since Harnack curves realize the maximal number of real components curves of the pencil may have, to each critical point of index 1 of a regular Harnack polynomial corresponds a "gluing" of real components in the pencil. These gluings are the subject of this section.

The main result of this section is gathered in the Lemma 2.6 of Chapter 4, Lemma 2.7 of Chapter 4 and Lemma 2.9 of Chapter 4 where we describe critical points of index 1 of Harnack polynomials.

1. Preliminaries

Given a regular polynomial $R(x_0, x_1, x_2) = x_0^m.r(x_1/x_0, x_2/x_0)$, for any real finite critical point (x_0, y_0) of $r(x, y)$, $r(x_0, y_0) = c_0$, there exists $\epsilon > 0$ sufficiently small such that $r^{-1}[c_0 - \epsilon, c_0 + \epsilon]$ contains no critical point other than (x_0, y_0).
We shall call *topological meaning* of (x_0, y_0) the homeomorphism type of the following triad of spaces $(W(x_0, y_0); \partial_0 W(x_0, y_0), \partial_1 W(x_0, y_0))$ with

$$W(x_0, y_0) = \cup_{c \in [c_0 - \epsilon, c_0 + \epsilon]} \{(x_0 : x_1 : x_2) \in \mathbf{C}P^2 | x_0^m (r(x_1/x_0, x_2/x_0) - c)) = 0\}$$

$$\partial_0 W(x_0, y_0) = \{(x_0 : x_1 : x_2) \in \mathbf{C}P^2 | x_0^m (r(x_1/x_0, x_2/x_0) - (c_0 - \epsilon)) = 0\}$$

$$\partial_1 W(x_0, y_0) = \{(x_0 : x_1 : x_2) \in \mathbf{C}P^2 | x_0^m (r(x_1/x_0, x_2/x_0) - (c_0 + \epsilon)) = 0\}$$

Let $B(x_0, y_0) \subset \mathbf{C}P^2$ be a small 4-ball around (x_0, y_0) globally invariant by complex conjugation such that $\partial_0 W(x_0, y_0) \cap B(x_0, y_0) \neq \emptyset$ and $\partial_1 W(x_0, y_0) \cap B(x_0, y_0) \neq \emptyset$

We shall call *local topological meaning* of (x_0, y_0) the homeomorphism type of the following triad of spaces
$(W(x_0, y_0) \cap B(x_0, y_0); \partial_0 W(x_0, y_0) \cap B(x_0, y_0), \partial_1 W(x_0, y_0) \cap B(x_0, y_0))$

In what follows, we shall associate to each critical point p of index 1 of a Harnack polynomial of degree m, two real branches of the Harnack curve of degree m involved in the local topological meaning of p.

Terminology

Recall that we distinguish two ways to perturb a crossing p of a singular curve $\mathcal{A}$ defined in the section 3 of Chapter 2 as *perturbation of type 1* and *perturbation of type 2* of p.

Thus, local topological meaning of a crossing (x_0, y_0) can be deduced from one topological space $\partial_1 W(x_0, y_0) \cap B(x_0, y_0)$ or $\partial_0 W(x_0, y_0) \cap B(x_0, y_0)$ and the type 1 or 2 of the perturbation of the point (x_0, y_0). Let us introduce the notations: $(\partial_1 W(x_0, y_0) \cap B(x_0, y_0))_i$ and $(\partial_0 W(x_0, y_0) \cap B(x_0, y_0))_i$ where the subscript i stands for the type of the perturbation of the crossing.

Although any perturbation of the union of two projective lines leads to a conic with orientable real part, the perturbation of a singular curve of degree $m > 2$ of which singular points are crossings may lead to a curve with non-orientable part. Nonetheless, in case of a Harnack curve $\mathcal{H}_m$, any deformation of a crossing which appears in a maximal simple deformation of $\mathcal{H}_m$ (see Proposition 3.6 of Chapter 2 and Proposition 3.7 of Chapter 2) agrees with a complex orientation of its real part $\mathbf{R}\mathcal{H}_m$. Besides, it is easy to deduce from relative orientation and location of real branches of $\mathbf{R}\mathcal{H}_m$ that any such crossing is deformed by a perturbation of type 1.

2. Critical points and recursive construction of Harnack curves

Introduction. Let $\mathcal{H}_m$ be the Harnack curve of degree m. In what follows, for any $m > 0$, we shall consider only Harnack polynomials $B_m(x_0, x_1, x_2) = x_0^m.b_m(x_1/x_0, x_2/x_0)$ of degree m and type $\mathcal{H}^0$. Since no confusion is possible, we shall call such polynomials Harnack polynomials. As already introduced in the preceding section (see proof of propositions 3.6 of Chapter 2 and 3.1 of Chapter 2), one can associate critical points of index 1 of B_m with crossings of curves $\mathcal{H}_{m-1} \cup L$.

Formally, we shall say that:

Definition 2.1. *Let a be a crossing of the curve $\mathcal{H}_{m-i} \cup L$. Given a disc $D(a, \epsilon)$ (in the Fubini-Study metric) of radius ϵ around of a in $\mathbf{R}P^2$.*

Let $U(a, \epsilon)$ be the neighborhood of $\mathcal{H}_{m-i+1}$ in $\mathbf{C}P^2$ deduced from the action of $U^3_{\mathbf{C}}$ on $U(a)$:

$$U(a, \epsilon) = \{z = \langle u, p \rangle = (u_0.p_0 : u_1.p_1 : u_2.p_2) \in \mathbf{C}P^2 \mid u = (u_0, u_1, u_2) \in U^3_{\mathbf{C}}, p \in U(a)\}$$
.

A critical point (x_0, y_0) of index 1 of B_m is associated with a crossing a of the curve $\mathcal{H}_{m-i} \cup L$ if there exists ϵ_0 such that:

(1) $a, (x_0, y_0) \in U(a, \epsilon_0)$
(2) *the perturbation on the real part of* $\mathcal{H}_m$ *involved in the local topological meaning of* (x_0, y_0) *is a deformation on* $\mathbf{C}\mathcal{H}_m \cap U(a, \epsilon_0)$

As already noticed corollary 3.5 of Chapter 2 of the previous section 3 of Chapter 2, without loss of generality, one can consider that Harnack curves are obtained via the Patchworking method.

Hence, for sake of simplicity, we shall consider the T-Harnack curves introduced in the Chapter 1. For the patchworking construction of Harnack we can give the following of version definition 2.1.

Recall that $\rho^m : \mathbf{R}_+T_m \times U^2_{\mathbf{C}} \to \mathbf{C}T_m \approx \mathbf{C}P^2$ and its restriction $\rho^m|_{\mathbf{R}_+T_m \times U^2_{\mathbf{R}}} : \mathbf{R}_+T_m \times U^2_{\mathbf{R}} \to \mathbf{R}T_m \approx \mathbf{R}P^2$ denote the natural surjections.

Recall (see definition 1.8 of Chapter 0) that given Γ a face of the triangulation of T_m and $D(p, \epsilon) \in \mathbf{R}_+T^0_m$ an (euclidian) open 2-disc such that the moment map $\mu : \mathbf{C}T_m \to T_m$ maps $D(p, \epsilon)$ to a two-disc $\mu(D(p, \epsilon)$ which contains Γ and intersects only the face Γ of the triangulation of T_m; we call $U(p) = \rho^m(D(p, \epsilon) \times U^2_{\mathbf{C}})$ the ϵ-neighborhood of $\mathbf{C}\mathcal{H}_m$ in $\mathbf{C}T_m \approx \mathbf{C}P^2$ defined from Γ^0; we call ϵ-tubular neighborhood of $\mathbf{C}\mathcal{H}_m$ defined from Γ^0 the ϵ-tubular neighborhood of $\mathbf{C}\mathcal{H}_m$ in $U(p)$.

Definition 2.2. *Let* Γ *be a face of the triangulation of* T_m *and* $U(p)$ *be the* ϵ*-neighborhood of* $\mathbf{C}\mathcal{H}_m$ *defined from* Γ^0*. A critical point* (x_0, y_0) *of index 1 of* B_m *is associated with a crossing* a *of the curve* $\mathcal{H}_{m-i} \cup L$ *if :*

(1) $a, (x_0, y_0) \in U(p)$
(2) *the perturbation on the real part of* $\mathcal{H}_m$ *involved in the local topological meaning of* (x_0, y_0) *is a deformation on* $\mathbf{C}\mathcal{H}_m \cap U(p)$

Definition 2.3. *Let* p_m *(resp* p_{m+j}*,* $j > 0$*) be a critical point of index 1 of a Harnack polynomial of degree* m *(resp* $m + j$*). We say that* p_{m+j} *is equivalent to* p_m *if there exist small conj-equivariant open* 4*-balls* $B(p_{m+j})$ *and* $B(p_m)$ *around* p_{m+j} *and* p_m *with the following properties:*

(1) $B(p_{m+j}) \subset B(p_m)$
(2) *The triad of spaces*

$$(W(p_{m+j} \cap B(p_{m+j}); \partial_0 W(p_{m+j}) \cap B(p_{m+j}), \partial_1 W(p_{m+j}) \cap B(p_{m+j}))$$

is the local topological meaning of p_{m+j}.

The triad of spaces

$$(W(p_m \cap B(p_m); \partial_0 W(p_m) \cap B(p_m), \partial_1 W(p_m) \cap B(p_m))$$

is the local topological meaning of p_{m+j}.

(3) *Local topological meanings of* p_{m+j} *and* p_m *are homeomorphic.*

Remark 2.4. Let $B_m(x_0, x_1, x_2)$ be a T-Harnack polynomial. It is an easy consequence of the T-construction of Harnack curves, that one can define 4-ball of $\mathbf{C}P^2$ around any critical point of $B_m(x_0, x_1, x_2)$ as usual 4-ball of $(\mathbf{C}^*)^2$. Indeed, none of the critical points of $B_m(x_0, x_1, x_2)$ belongs to the coordinates axes.

We shall denote p_{m+j} is equivalent to p_m by $p_{m+j} \approx p_m$.

DEFINITION 2.5. *Given a a crossing of $\mathcal{H}_{m-1} \cup L$, denote by p_l the critical point of index 1 of a Harnack polynomial of degree $l \geq m$ associated with a.*

(1) *We say that p_m is the simple point of $\mathcal{H}_m$ associated with a.*
(2) *If for any $j \geq 1$, p_{m+j} is equivalent to p_m, we say that a simple point of $\mathcal{H}_{m+j}$ with $j \geq 0$ is associated with a.*
(3) *If for any integers l,l' ,$0 < l \neq l' \leq k$, p_{m+l} is not equivalent to $p_{m+l'}$; for any $l \leq k$, we say that a l-point $(p_m, p_{m+1}, ..., p_{m+l})$ of $\mathcal{H}_{m+l}$ is associated with a.*
(4) *If k is the smallest integer such that: for any integers l, l', $0 < l \neq l' \leq k$, p_{m+l} is not equivalent to $p_{m+l'}$; for any $l \geq k$, we say that a k-point $(p_m, p_{m+1}, ..., p_{m+k})$ of $\mathcal{H}_{m+l}$ is associated with a.*

Critical points of index 1 of Harnack polynomials. We shall now proceed to the study of critical points of index 1 of Harnack polynomials. We shall work with the notations introduced in the chapter 2. Given $B_m(x_0, x_1, x_2) = x_0^m.b_m(x_1/x_0, x_2/x_0)$ a Harnack polynomial of type $\mathcal{H}^0$, we denote by S_m the set of critical points of index 1 of $b_m(x, y)$, and by S_m^- (resp, S_m^+) the subset of S_m consisting respectively of critical points of index 1 with negative (resp, positive) critical value.

In case $m = 2k$, we distinguish the two subsets S'_{2k}, $S"_{2k}$ of S_{2k}^+ with the properties $S'_{2k} \cup S"_{2k} = S_{2k}^+$ $S'_{2k} \cap S"_{2k} = \emptyset$. Given $B_{2k}(x_0, x_1, x_2)$ a Harnack polynomial of type $\mathcal{H}^0$ and $b_{2k}(x_1/x_0, x_2/x_0)$ its affine associated polynomial. The set S'_{2k} is constituted by the $c'_1(B_{2k})$ critical points (x_0, y_0) of index 1 $b_{2k}(x_0, y_0) = c_0$, $c_0 > 0$ with the property that as c increases from $c_0 - \epsilon$ to $c_0 + \epsilon$ the number of real connected components of $\mathcal{A}_c = \{(x_0 : x_1 : x_2) \in \mathbf{R}P^2 | x_0^{2k}.(b_{2k} - c) = 0\}$ intersecting the line at infinity increases by 1. The subset $S"_{2k}$ denotes the complementary set of S'_{2k} inside S_{2k}^+. These sets were already under consideration in propositions 1.4 of Chapter 2, 3.6 of Chapter 2 and theorem 0.10 of Chapter 3.

Let us start by the Lemma 2.6 of Chapter 4 in which we prove that critical points of negative critical value of Harnack polynomials of type $\mathcal{H}^0$ are simple points in the recursive construction of Harnack curves.

LEMMA 2.6. *Let $\mathcal{H}_m$ be a Harnack curve of degree m obtained via the patchworking method and given by a Harnack polynomial of type $\mathcal{H}^0$.*

(1) *In case of odd m, let Γ be a face of the triangulation of T_m which belongs to l_{m-1}.*
(2) *In case of even m, let Γ given by vertices $(c, d+1)(c+1, d)$ with $c = 0$ or $d = 0$.*

Let $U(p)$ be the ϵ-neighborhood of $\mathbf{C}\mathcal{H}_m$ in $\mathbf{C}P^2$ defined from Γ^0. Denote a the unique crossing of $\mathcal{H}_{m-1} \cup L$ which belongs to $U(p)$. Denote p_m the critical point of $\mathcal{H}_m$ associated with a, and let $B(p_m) \subset U(p)$ a conj-equivariant 4-ball around p_m. Then,

(1) *the critical point p_m of $\mathcal{H}_m$ belongs to S_m^-*
Moreover, $\mathcal{H}_m \cap B(p_m) \approx (\partial_1 W(p_m) \cap B(p_m))_1$
(2) *a simple-point of curves $\mathcal{H}_{m+j}$, $j \geq 0$, is associated with a*

proof:

The proof is based on the Petrovskii's theory and propositions 1.4 of Chapter 2, 3.6 of Chapter 2 and 3.1 of Chapter 2. Our argumentation is similar to the one used in the proof of proposition 3.1 of Chapter 2.

Assume Γ given by vertices $(c, d+1)(c+1, d)$. For any $j \geq 0$, given $B_{m+j} = x_0^{m+j} b_{m+j}(x_1/x_0, x_2/x_0)$ the Harnack polynomial of degree $m+j$, we shall denote b^S_{m+j} the truncation of b_{m+j} on the monomials $x^c y^d$,$x^c y^{d+1}$ $x^{c+1}y^d$, $x^{c+1}y^{d+1}$.

We shall use local description of Harnack curves, $\mathcal{H}_{m+j}$, $j \geq 0$, inside $U(p)$ provided by the patchworking theory.

Recall that for any $j \geq 0$, there exists an homeomorphism $\tilde{h} : \mathbf{C}\mathcal{H}_{m+j} \cap U(p) \to \{(x,y) \in (\mathbf{C}^*)^2 | b^S_{m+j} = 0\} \cap U(p)$ such that $\tilde{h}(a) = (x_0, y_0)$ is a critical point of b^S_{m+j}.

We shall divide our proof in two parts. In the first part, we shall consider the curve $\mathcal{H}_m$ and study local topological meaning of crossings of $\mathcal{H}_{m-1} \cup L$. Then, we shall consider curves $\mathcal{H}_{m+j}$, $j \geq 1$.

(1) Let us distinguish the cases m even and m odd.

(1).a In case of even $m = 2k$

Let $\Gamma \in l_{2k-1}$ given by vertices $(c, d+1)(c+1, d)$ with $c = 0$ or $d = 0$. (Namely, Γ has vertices $(0, 2k-1), (1, 2k-2)$ or $(2k-1, 0), (2k-2, 1)$.) From the patchworking theory, (see corollary 2.4 of Chapter 1 of the chapter 1) $\tilde{h}(a) = (x_0, y_0)$ $x_0 > 0, y_0 < 0$ is a critical point of $b^S_{2k}(x,y)$.

Keeping the notations introduced in the chapter 2, set $b_{2k} = x_{2k;t}$ with $t \in]0, t_{2k}[$. Then, for fixed $t \in]0, t_{2k}[$, let a_t be the crossing of $\mathcal{H}_{2k-1} \cup L$ contained in $U(p)$.

Therefore, $\tilde{h}(a_t) = (x_0, y_0)_t$ $x_0 > y_0 < 0$ is a critical point of $x^S_{2k,t}(x,y)$.

Set $x^S_{2k;t}(x,y) = l_t(x,y) + t.k_t(x,y)$ with
$l_t(x,y) = a_{c,d} t^{\nu(c,d)} x^c y^d + a_{c+1,d} t^{\nu(c,d+1)} x^{c+1} y^d$;
$k_t(x,y) = a_{c,d+1} t^{\nu(c+1,d)-1} x^c y^{d+1} + -a_{c+1,d+1} t^{\nu(c+1,d+1)-1} x^{c+1} y^{d+1}$.

Then, it is easy to see that, up to modify the coefficients $a_{c,d}, a_{c,d+1}, a_{c+1,d}, a_{c+1,d}$ if necessary, the point $\tilde{h}(a_t) = (x_1, y_1)_t$ is a critical point of the function $\frac{l_t(x,y)}{k_t(x,y)}$ with negative critical value $\geq -t > -t_{2k-1}$.

On this assumption, it follows from the equalities
$l_t(x,-y) = -l_t(x,y)$, $k_t(x,-y) = k_t(x,y)$,
$\frac{\partial l_t}{\partial x}(x,-y) = -\frac{\partial l_t}{\partial x}(x,y)$, $\frac{\partial l_t}{\partial y}(x,-y) = \frac{\partial l_t}{\partial y}(x,y)$,
$\frac{\partial k_t}{\partial x}(x,-y) = \frac{\partial k_t}{\partial x}(x,y)$, $\frac{\partial k_t}{\partial y}(x,-y) = -\frac{\partial k_t}{\partial y}(x,y)$
that $(x_1, -y_1)_t$ is a critical point of the function $\frac{l_t(x,y)}{k_t(x,y)}$ with positive critical value $\leq t < t_{2k-1}$. Therefore, up to modify coefficients of b^S_{2k}, $(x_0, -y_0)$ is a critical point with negative critical value of b^S_{2k}. Obviously, (see [**16**]), such modification does not affect the order and the topological structure of b^S_{2k}. Therefore, it does not change local topological meaning of critical points of the Harnack polynomial B_{2k}.
We shall now proceed to the study of the Petrovskii's pencil of curves over $\mathcal{H}_{2k}$.

Let d_{2k} be the unique polynomial such that $b_{2k} = b^S_{2k} + d_{2k}$. Consider curves of the Petrovskii's pencil over $\mathcal{H}_{2k}$ outside $d_{2k} = 0$, as level curves of the function $\frac{b^S_{2k} - c}{d_{2k}}$.

Inside $b^S_{2k} - c = 0 \backslash d_{2k} = 0$, these curves have critical points the singular points of $b^S_{2k} - c = 0$.

Bringing together Petrovskii's theory and the implicit function theorem applied to one-parameter polynomial
$x_0^{2k}.(b_{2k} - c) = x_0^{2k}.(b^S_{2k} + d_{2k} - c)$ with parameter c, it follows that a point p_{2k} with $b_{2k}(p_{2k}) = c_0 < 0$ is associated with a.

When $c = c_0$, one positive oval touches an other positive oval.

(1).b In case of odd $m = 2k+1$, one has to distinguish whether Γ intersects or not the coordinates axes.

(1).b.1

Let $\Gamma \in l_{2k}$ given by vertices $(c, d+1)(c+1, d)$ with
$c = 0$ or $d = 0$. (Namely, Γ has vertices $(0, 2k), (1, 2k-1)$ or $(2k, 0), (2k-1, 1)$.)

According to the patchworking theory, (see corollary 2.4 of Chapter 1 of the first section) $\tilde{h}(a) = (x_0, y_0)$ with $x_0.y_0 > 0$ is a critical point of $b^S_{2k+1}(x, y)$. Set $\tilde{h}(a) = (x_0, y_0) x_0 > 0, y_0 > 0$.

On this assumption, following the previous argumentation, up to modify coefficients of b^S_{2k+1} if necessary (without changing the order and the topological structure of b_{2k+1}) $(x_0, -y_0)$ is a critical point with negative critical value of b^S_{2k+1}.

Therefore, according to Petrovskii's theory, from an argumentation similar to the one given in even case, a point p_{2k+1} with $b_{2k+1}(p_{2k+1}) = c_0 < 0$ is associated with a.

When $c = c_0$, a positive oval touches the one-side component of the curve.

(1).b.2

We shall now consider faces which do not intersect the coordinates axes. Our argumentation is a slightly modified version of the previous one. We shall study the situation inside ϵ-neighborhood defined from two faces together. Let $\Gamma \in l_{2k}$ given by vertices $(c, d+2), (c+1, d+1)$ with c odd, $c \neq 0$ and $d \neq 0$ and $\Gamma' \in l_{2k}$ given by vertices $(c+2, d), (c+1, d+1)$.

Consider the convex polygon $K \subset T_{2k+1}$ with vertices
$(c+1, d), (c+2, d), (c+2, d+1), (c+1, d+2), (c, d+2), (c, d+1)$. It is contained in the triangle T_{2k+1} and triangulated by the triangulation τ of T_{2k+1}. Denote $U(p)$ the ϵ-neighborhood of $\mathbf{C}\mathcal{H}_{2k+1}$ defined from Γ^0. Denote $U(p')$ the ϵ-neighborhood of $\mathbf{C}\mathcal{H}_{2k+1}$ defined from ${\Gamma'}^0$. Let a, (resp, a') be the crossing of $\mathcal{H}_{2k} \cup L$ which belongs to $U(p)$, (resp, $U(p')$). Denote b^S_{2k+1} (resp, $b^{S'}_{2k+1}$) the truncation of b_{2k+1} on the monomials $x^c y^{d+1}$,$x^c y^{d+2}$ $x^{c+1} y^{d+2}$, $x^{c+1} y^{d+1}$ (resp, $x^{c+1} y^d$,$x^{c+1} y^{d+1}$ $x^{c+2} y^{d+1}$, $x^{c+2} y^d$.)

As previously, consider homeomorphisms
$\tilde{h} : \mathbf{C}\mathcal{H}_{2k+1} \cap U(p) \to \{(x, y) \in (\mathbf{C}^*)^2 | b^S_{2k+1} = 0\} \cap U(p)$ such that $\tilde{h}(a) = (x_0, y_0)$ $x_0 > 0, y_0 > 0$ is a critical point of b^S_{2k+1}

and $\tilde{h}' : \mathbf{C}\mathcal{H}_{2k+1} \cap U(p') \to \{(x, y) \in (\mathbf{C}^*)^2 | b^{S'}_{2k+1} = 0\} \cap U(p')$ such that $\tilde{h}'(a') = (x'_0, y'_0)$ $x_0 > 0, y_0 > 0$ is a critical point of $b^{S'}_{2k+1}$.

Keeping the notations introduced in the chapter 2, set $b_{2k+1} = x_{2k;t}$, $t \in]0, t_{2k+1}[$.

For fixed $t \in]0, t_{2k+1}[$, let a_t (resp a'_t) be the crossing of $\mathcal{H}_{2k} \cup L$ contained in $U(p)$ (resp $U(p')$). Set $x^K_{2k+1;t}(x, y) = l_t(x, y) + t.k_t(x, y)$ with

$l_t(x,y) = a_{c+1,d}t^{\nu(c,d)}x^{c+1}y^d + a_{c+2,d}t^{\nu(c+2,d)}x^{c+2}y^d +$
$a_{c+1,d+2}t^{\nu(c,d)}x^{c+1}y^{d+2} + a_{c+2,d+2}t^{\nu(c+2,d)}x^{c+2}y^{d+2}$;
$k_t(x,y) = a_{c,d+1}t^{\nu(c+1,d)-1}x^c y^{d+1} - a_{c+1,d+1}t^{\nu(c+1,d+1)-1}x^{c+1}y^{d+1} +$
$a_{c+2,d+1}t^{\nu(c+1,d)-1}x^{c+2}y^{d+1}$

Then, it is easy to see that, up to modify the coefficients $a_{i,j}$ if necessary, points $\tilde{h}(a_t) = (x_0, y_0)_t$ and $\tilde{h}'(a'_t) = (x'_0, y'_0)_t$ are critical points of the function $\frac{l_t(x,y)}{k_t(x,y)}$ with negative critical value $\geq -t > -t_{2k-1}$. On this assumption, it follows from the equalities
$l_t(x,-y) = -l_t(x,y)$, $k_t(x,-y) = k_t(x,y)$,
$\frac{\partial l_t}{\partial x}(x,-y) = -\frac{\partial l_t}{\partial x}(x,y)$, $\frac{\partial l_t}{\partial y}(x,-y) = \frac{\partial l_t}{\partial y}(x,y)$,
$\frac{\partial k_t}{\partial x}(x,-y) = \frac{\partial k_t}{\partial x}(x,y)$, $\frac{\partial k_t}{\partial y}(x,-y) = -\frac{\partial k_t}{\partial y}(x,y)$ that points $(x_0,-y_0)_t$ and $(x'_0,-y'_0)_t$ are critical point of the function $\frac{l_t(x,y)}{k_t(x,y)}$ with positive critical value $\leq t < t_{2k-1}$. Therefore, up to modify the coefficients of b^K_{2k+1}, $(x_0,-y_0)$ and $(x'_0,-y'_0)$ are critical points with negative critical value of b^K_{2k+1}. Obviously, (see [**16**]), such modification does not modify the order and the topological structure of b^K_{2k+1}. Therefore, it does not change local topological meanings of critical points of the Harnack polynomial B_{2k+1}.
We shall now proceed to the study of the Petrovskii's pencil of curves over $\mathcal{H}_{2k+1}$. Let $U(K^0)$ be the subset $\rho^{2k+1}(\mathbf{R}_+K^0 \times U^2_{\mathbf{C}})$ of $\mathbf{C}P^2$.
Obviously, $U(p) \cup U(p') \subset U(K^0)$. According to patchworking theory, the truncation b^K_{2k+1} of b_{2k+1} is ϵ-sufficient for b_{2k+1} in $U(K^0)$. Therefore,
$\tilde{h} : \mathbf{C}\mathcal{H}_{2k+1} \cap U(p) \to \{(x,y) \in (\mathbf{C}^*)^2 | b^S_{2k+1} = 0\} \cap U(p)$
and $\tilde{h} : \mathbf{C}\mathcal{H}_{2k+1} \cap U(p') \to \{(x,y) \in (\mathbf{C}^*)^2 | b^{S'}_{2k+1} = 0\} \cap U(p')$ extend to the homeomorphism $\tilde{h} : \mathbf{C}\mathcal{H}_{2k+1} \cap U(K^0) \to \{(x,y) \in (\mathbf{C}^*)^2 | b^K_{2k+1} = 0\} \cap U(K^0)$ such that $\tilde{h}(a) = (x_0,y_0)$ $x_0.y_0 > 0$ $\tilde{h}(a') = (x'_0,y'_0)$ $x'_0.y'_0 > 0$. Let d_{2k+1} be the unique polynomial such that $b_{2k+1} = b^K_{2k+1} + d_{2k+1}$. Consider curves of the Petrovskii's pencil over $\mathcal{H}_{2k+1}$ outside $d_{2k+1} = 0$, as level curves of the function $\frac{b^K_{2k+1} - c}{d_{2k+j}}$.

From an argumentation similar to the previous one, according to the Petrovskii's theory, it follows that: a point p_{2k+1} with $b_{2k+1}(p_{2k+1}) = c_0 < 0$ is associated with a and a point p'_{2k+1}, $b_{2k+1}(p'_{2k+1}) = c'_0 < 0$ is associated with a'. When $c = c_0$, the one-side component of the curve touches itself. When $c = c'_0$, the one-side component of the curve touches itself.

(2) Let a be one singular point of the curve $\mathcal{H}_{m-1} \cup L$ under consideration in the part 1 of the proof and p_m be the critical point of the Harnack polynomial of degree m associated with a. We shall now prove that for any such crossing a and any $j \geq 1$, there exists a critical point p_{m+j} of a Harnack polynomial B_{m+j} associated with a. The point p_{m+j} is equivalent to the critical point p_m of B_m.

Let $j \geq 1$ and B_{m+j} be a Harnack polynomial of degree $m+j$

As previously, consider the homeomorphism
$\tilde{h} : \mathbf{C}\mathcal{H}_{m+j} \cap U(p) \to \{(x,y) \in (\mathbf{C}^*)^2 | b^S_{m+j} = 0\} \cap U(p)$.

(2).a

Assume Γ intersects the coordinates axes ($\Gamma \in l_{2k-1}$ or $\Gamma \in l_{2k}$) Let d_{m+j} be the unique polynomial such that $b_{m+j} = b^S_{m+j} + d_{m+j}$. Consider curves of the Petrovskii's pencil over $\mathcal{H}_{m+j}$ outside $d_{m+j} = 0$, as level curves of the function

$\frac{b^S_{m+j}-c}{d_{m+j}}$. Inside $b^S_{m+j} - c = 0 \backslash d_{m+j} = 0$, these curves have critical points the singular points of $b^S_{m+j} - c = 0$.

From an argumentation similar to the previous one, according to the Petrovskii's theory (Petrovskii's pencil over $\mathcal{H}_{m+j}$), it follows easily that for any $j \geq 1$, a point p_{m+j} is associated with a and is equivalent to the point p_m.

(2).b Assume Γ does not intersect the coordinates axes ($\Gamma \in l_{2k}$). Consider as previously the convex polygon K with vertices $(c+1, d), (c+2, d), (c+2, d+1), (c+1, d+2), (c, d+2), (c, d+1)$. Let d_{m+j} be the unique polynomial such that $b_{m+j} = b^K_{m+j} + d_{m+j}$.

Consider curves of the Petrovskii's pencil over $\mathcal{H}_{m+j}$ outside $d_{m+j} = 0$, as level curves of the function $\frac{b^K_{m+j}-c}{d_{m+j}}$. From an argumentation similar to the previous one, according to the Petrovskii's theory, it follows that for any $j \geq 1$, a point p_{m+j} is associated with a and is equivalent to the point p_m. Q.E.D.

Then, consider the case when a critical point p_m of S^+_m is associated to a crossing of $\mathcal{H}_{m-1} \cup L$. Such critical point appears the first time as critical point of a Harnack polynomial of degree 4.

Recall (see the Proposition 1.4 of Chapter 2) that for a Harnack polynomial $B_{2k}(x_0, x_1, x_2) = x_0^{2k}.b_{2k}(x_1/x_0, x_2/x_0)$ of type $\mathcal{H}^0$ we denote S'_{2k} the set constituted by the $c'_1(B_{2k})$ critical points (x_0, y_0) of index 1 $b_{2k}(x_0, y_0) = c_0$, $c_0 > 0$ with the property that as c increases from $c_0 - \epsilon$ to $c_0 + \epsilon$ the number of real connected components intersecting the line at infinity increases by 1. In the Lemma 2.7 of Chapter 4, we shall study how critical points S'_{2k} of a Harnack polynomial of type $\mathcal{H}^0$ and degree $2k$ vary in the recursive construction of Harnack curves.

LEMMA 2.7. *Let $\mathcal{H}_{2k}$ be the Harnack curve of degree $2k$ obtained via the patchworking method and given by a Harnack polynomial of type $\mathcal{H}^0$. Let Γ be a face of the triangulation of T_{2k} contained into the line l_{2k-1} and given by vertices $(c+1, d), (c, d+1)$ with c and d odd. Let $U(p)$ be the ϵ-neighborhood of $\mathbf{C}\mathcal{H}_m$ defined from Γ^0. Denote a the unique crossing of $\mathcal{H}_{2k-1} \cup L$ such that $a \in U(p)$.*

Then, a 3-point $(p_{2k}, p_{2k+1}, p_{2k+2})$ of Harnack curves $\mathcal{H}_{2k+j}$,$j \geq 3$, is associated with the point a.

(1) *The point $p_{2k} \in S'_{2k} \subset S^+_{2k}$. Besides, there exists a positive oval $\mathcal{O}$ such that the point $p_{2k} \in S^+_{2k}$ has the property:*
$$\mathcal{O} \notin \partial_1 W(p_{2k}), \mathcal{O} \in \partial_0 W(p_{2k})$$
Moreover, given $B(p_{2k})$ a conj-equivariant 4-ball around p_{2k},
$$\mathcal{H}_{2k} \cap B(p_{2k}) \approx (\partial_0 W(p_{2k}) \cap B(p_{2k}))_1$$
(2) *There exists a negative oval $\mathcal{O}$ such that the point $p_{2k+1} \in S^+_{2k+1}$ has the property:*
$$\mathcal{O} \in \partial_0 W(p_{2k+1}), \mathcal{O} \notin \partial_1 W(p_{2k+1})$$
Moreover, given $B(p_{2k+1})$ a conj-equivariant 4-ball around p_{2k+1},
$$\mathcal{H}_{2k+1} \cap B(p_{2k+1}) \approx (\partial_0 W(p_{2k+1}) \cap B(p_{2k+1}))_1$$

(3) *There exists a positive oval* $\mathcal{O}$ *of* $\mathcal{H}_{2k+2}$ *such that the point* $p_{2k+2} \in S^-_{2k+2}$ *has the property:*

$$\mathcal{O} \in \partial_1 W(p_{2k+2}), \mathcal{O} \notin \partial_0 W(p_{2k+2})$$

Moreover, given $B(p_{2k+2})$ *a conj-equivariant* 4*-ball around* p_{2k+2}*,*

$$\mathcal{H}_{2k+2} \cap B(p_{2k+2}) \approx (\partial_1 W(p_{2k+2}) \cap B(p_{2k+2}))_1$$

DEFINITION 2.8. We shall call 3*-point of the first kind* a 3-point verifying all assumptions of Lemma 2.7 of Chapter 4.

proof:

As in the proof of Lemma 2.6 of Chapter 4, we shall use local description of Harnack curves $\mathcal{H}_{m+j}$, $j \geq 0$, inside $U(p)$ provided by the patchworking theory.

Recall that for any $j \geq 0$, there exists an homeomorphism $\tilde{h} : \mathbf{C}\mathcal{H}_{m+j} \cap U(p) \to \{(x, y) \in (\mathbf{C}^*)^2 | b^S_{m+j} = 0\} \cap U(p)$ such that $\tilde{h}(a) = (x_0, y_0)$ $x_0.y_0 < 0$ is a critical point of b^S_{m+j}.

Our proof uses the results of chapter 2 and in particular proposition 1.4 of Chapter 2, and proposition 3.6 of Chapter 2. We shall work with notations introduced in the chapter 2.

(1) Let $\mathcal{H}_{2k}$ the Harnack curve of degree $2k$ ($k > 2$) and B_{2k} be a Harnack polynomial of degree $2k$. According to proposition 3.6 of Chapter 2, a point $p_{2k} \in S'_{2k}$ is associated with a.

Set $b_{2k}(p_{2k}) = c_0 > 0$, then when $c = c_0$, the non-empty positive oval touches a positive outer oval.

(2) Let $\mathcal{H}_{2k+1}$ be the Harnack curve of degree $2k+1$ and B_{2k+1} be a Harnack polynomial of degree $2k+1$.

Consider the convex polygon $K \subset T_{2k+1}$ with vertices $(c, d), (c, d+2), (c+1, d+2), (c+2, d), (c+2, d+1)$. It is contained in the triangle T_{2k+1} and triangulated by the triangulation τ of T_{2k+1}.

Let $U(K^0)$ be the subset $\rho^{2k+1}(\mathbf{R}_+ K^0 \times U^2_{\mathbf{C}})$ of $\mathbf{C}P^2$. Obviously, $U(p) \subset U(K^0)$.

Denote b^K_{2k+1} the truncation of b_{2k+1} on the monomials $x^c y^d$, $x^c y^{d+1}$, $x^c y^{d+2}$, $x^{c+1} y^d$, $x^{c+1} y^{d+1}$, $x^{c+1} y^{d+2}$, $x^{c+2} y^d$, $x^{c+2} y^{d+1}$. (Namely, $b^K_{2k+1} = b^S_{2k+1} + a_{c,d+2} x^c y^{d+2} + a_{c+1,d+2} x^{c+1} y^{d+2} + a_{c+2,d} x^{c+2} y^d + a_{c+2,d+1} x^{c+2} y^{d+1}$ with $a_{c,d+2} > 0, a_{c+1,d+2} > 0, a_{c+2,d} > 0, a_{c+2,d+1} > 0$.

According to the patchworking theory, the truncation b^K_{2k+1} of b_{2k+1} is ϵ-sufficient for b_{2k+1} in $U(K^0)$.

Therefore, $\tilde{h} : \mathbf{C}\mathcal{H}_{2k+1} \cap U(p) \to \{(x, y) \in (\mathbf{C}^*)^2 | b^S_{2k+1} = 0\} \cap U(p)$ extends to the homeomorphism $\tilde{h} : \mathbf{C}\mathcal{H}_{2k+1} \cap U(K^0) \to \{(x, y) \in (\mathbf{C}^*)^2 | b^K_{2k+1} = 0\} \cap U(K^0)$ such that $\tilde{h}(a) = (x_0, y_0)$ $x_0.y_0 < 0$.

According to proposition 3.1 of Chapter 2, one can assume that $(x_0, -y_0)$, $x_0.-y_0 > 0$, $x_0 < 0, -y_0 < 0$, is a critical point of $b^S_{2k+1} = b^S_{2k}$ with positive critical value.

Thus, up to modify coefficients of the polynomials b^K_{2k+1} (without changing the order and the topological structure of b_{2k+1}), the point $(x_0, -y_0)$ is a critical point of b^K_{2k+1} with $b_{2k+1}(x_0, -y_0) > 0$.

Hence, we shall proceed to the study of the Petrovskii's pencil of curves over $\mathcal{H}_{2k+1}$. Let d_{2k+1} be the unique polynomial such that $b_{2k+1} = b^K_{2k+1} + d_{2k+1}$. Consider curves of the Petrovskii's pencil over $\mathcal{H}_{2k+1}$ outside $d_{2k+1} = 0$, as level curves of the function $\frac{b^K_{2k+1}-c}{d_{2k+1}}$.

Inside $b^K_{2k+1} - c = 0 \backslash d_{2k+1} = 0$, these curves have critical points the singular points of $b_{2k+1} - c = 0$.

Bringing together Petrovskii's theory and the implicit function theorem applied to one-parameter polynomial $x_0^{2k+1}.(b_{2k+1} - c)$ with parameter c, it follows that a point p_{2k+1} with $b_{2k+1}(p_{2k+1}) = c_0 > 0$ is associated with a.

When $c = c_0$, the one-side component of the curve $x_0^{2k+1}.(b_{2k+1} - (c_0)$ touches itself. Besides, according to Petrovskii's Lemma 2 (3.a) and Lemma 3 as c increases from c_0 to $c_0 + \epsilon$ one negative oval $\mathcal{O}^-$ (of the curve $x_0^{2k+1}.(b_{2k+1} - (c_0 - \epsilon) = 0)$ disappears. (In the pencil of curves, $\mathcal{O}^-$ as oval of $\mathcal{H}_{2k}$ has been created in the construction of $\mathcal{H}_{2k+1}$ from $\mathcal{H}_{2k} \cup L$)

(3) Let $\mathcal{H}_{2k+2}$ be the Harnack curve of degree $2k+2$ and B_{2k+2} be a Harnack polynomial of degree $2k+2$.

Consider J the square with vertices $(c, d), (c, d+2), (c+2, d), (c+2, d+2)$. It is contained in the triangle T_{2k+2} and triangulated by the triangulation τ of T_{2k+2}. Let $U(J^0)$ be the subset $\rho^{2k+2}(\mathbf{R}_+ J^0 \times U^2_{\mathbf{C}})$ of $\mathbf{C}P^2$. Obviously, $U(p) \subset U(K^0) \subset U(J^0)$.

Denote b^J_{2k+2} the truncation of b_{2k+2} on the monomials $x^c y^d$, $x^c y^{d+1}$, $x^c y^{d+2}$, $x^{c+1} y^d$, $x^{c+1} y^{d+1}$, $x^{c+1} y^{d+2}$, $x^{c+2} y^d$, $x^{c+2} y^{d+1}$,$x^{c+2} y^{d+2}$. (Namely, $b^J_{2k+1} = b^K_{2k+1} +$ $+ a_{c+2,d+2} x^{c+2} y^{d+2}$ with $a_{c+2,d+2} > 0$, $b^J_{2k+1} = a_{c,d} x^{c,d} + a_{c+1,d} x^{c+1} y^d + + a_{c,d+1} x^{c,d+1} -$ $a_{c+1,d} x^{c+1} y^d + + a_{c,d+2} x^c y^{d+2} + a_{c+1,d+2} x^{c+1} y^{d+2} + a_{c+2,d} x^{c+2} y^d + a_{c+2,d+1} x^{c+2} y^{d+1}$ with $a_{c,d+2} > 0, a_{c+1,d+2} > 0, a_{c+2,d} > 0, a_{c+2,d+1} > 0$.

According to the patchworking theory, the truncation b^J_{2k+2} of b_{2k+2} is ϵ-sufficient for b_{2k+2} in $U(J^0)$. Therefore, $\tilde{h} : \mathbf{C}\mathcal{H}_{2k+2} \cap U(p) \to \{(x, y) \in (\mathbf{C}^*)^2 | b^S_{2k+2} = 0\} \cap U(p)$ extends to the homeomorphism $\tilde{h} : \mathbf{C}\mathcal{H}_{2k+2} \cap U(J^0) \to \{(x, y) \in (\mathbf{C}^*)^2 | b^J_{2k+2} = 0\} \cap U(J^0)$ such that $\tilde{h}(a) = (x_0, y_0)$ $x_0.y_0 < 0$.

It is easy to see that, up to modify coefficients of the polynomials b^J_{2k+2}, (without changing the order and the topological structure of b_{2k+2}), the point $(x_0, -y_0)$, $x_0 < 0, -y_0 < 0$ is a critical point of b^J_{2k+2} with $b_{2k+2}(x_0, -y_0) < 0$.

Hence, we shall proceed to the study of the Petrovskii's pencil of curves over $\mathcal{H}_{2k+2}$.

Denote d_{2k+2} the unique polynomial such that $b_{2k+2} = b^J_{2k+2} + d_{2k+2}$.

As previously, consider curves of the Petrovskii's pencil over $\mathcal{H}_{2k+2}$ outside $d_{2k+2} = 0$, as level curves of the function $\frac{b^J_{2k+2}-c}{d_{2k+2}}$.

Inside $(b^J_{2k+2} - c = 0) \backslash (d_{2k+2} = 0)$, these curves have critical points the singular points of $b^J_{2k+2} - c = 0$. Bringing together Petrovskii's theory, the implicit function theorem applied to one-parameter polynomial $x_0^{2k+2}.(b_{2k+2} - c) = x_0^{2k}.(b^J_{2k+2} + d_{2k+2} - c)$ with parameter c, it follows that a point p_{2k+2} with $b_{2k+2}(p_{2k+2}) = c_0 < 0$ is associated with a.

Let $b_{2k+2}(p_{2k+2}) = c_0$. When $c = c_0$, one positive oval touches another positive oval. Then, as c decreases from c_0 to $c_0 - \epsilon$, a positive oval disappears.

Let $\mathcal{H}_{2k+j}$ be the Harnack curve of degree $2k + j$, $j \geq 3$ and let B_{2k+j} be a Harnack polynomial of degree $m + j$

As previously, consider b^J_{2k+j} the truncation of b_{2k+j} on the monomials $x^c y^d$, $x^c y^{d+1}$, $x^{c+1} y^d$, $x^{c+1} y^{d+1}$.

According to patchworking theory, the truncation b^K_{2k+j} of b_{2k+j} is ϵ-sufficient for b_{2k+j} in $U(J^0)$.

Therefore,
$\tilde{h} : \mathbf{C}\mathcal{H}_{2k+j} \cap U(p) \to \{(x, y) \in (\mathbf{C}^*)^2 | b^S_{2k+j} = 0\} \cap U(p)$ extends to the homeomorphism
$\tilde{h} : \mathbf{C}\mathcal{H}_{2k+j} \cap U(K^0) \to \{(x, y) \in (\mathbf{C}^*)^2 | b^K_{2k+1} = 0\} \cap U(K^0)$ such that $\tilde{h}(a) = (x_0, y_0)$ $x_0.y_0 > 0$.

Let d_{2k+j} be the unique polynomial such that $b_{2k+j} = b^J_{2k+j} + d_{2k+j}$. Consider curves of the Petrovskii's pencil over $\mathcal{H}_{2k+j}$ outside $d_{2k+j} = 0$, as level curves of the function $\frac{b^J_{2k+j} - c}{d_{2k+j}}$. From an argumentation similar to the previous one, it follows that a point p_{2k+j} of S^-_{2k+j} equivalent to p_{2k+2} is associated with a.

Therefore, a *3-point* of curves $\mathcal{H}_{2k+j}$, $j \geq 3$, is associated with a.

Q.E.D

Recall (see propositiony 3.6 of Chapter 2) that given $B_{2k} = x_0^{2k}.b_{2k}(x_1/x_0, x_2/x_0)$ a Harnack polynomial of degree $2k$ and type $\mathcal{H}^0$, $S"_{2k}$ denotes $S^+_{2k} \backslash S'_{2k}$ the complementary set of S'_{2k} inside S^+_{2k}. In the Lemma 2.9 of Chapter 4, we shall study how critical points $S"_{2k}$ of a Harnack polynomial of type $\mathcal{H}^0$ and degree $2k$ vary in the recursive construction of Harnack curves.

Lemma 2.9. *Let $\mathcal{H}_{2k}$ be the Harnack curve of degree $2k$ obtained via the patchworking method and given by a Harnack polynomial of type $\mathcal{H}^0$. Let Γ be a face of the triangulation of T_{2k} contained into the line l_{2k-1} and given by vertices $(c + 1, d), (c, d + 1)$ with c and d even and strictly positive. Let $U(p)$ be the ϵ-neighborhood of $\mathbf{C}\mathcal{H}_m$ defined from Γ^0. Denote a the unique crossing of $\mathcal{H}_{2k-1} \cup L$ such that $a \in U(p)$.*

Then, a 3-point $(p_{2k}, p_{2k+1}, p_{2k+2})$ of Harnack curves $\mathcal{H}_{2k+j}$, $j \geq 3$, is associated with a.

(1) *There exists a negative oval $\mathcal{O}$ such that the point $p_{2k} \in S"_{2k} \subset S^+_{2k}$ has the property: $\mathcal{O} \in \partial_0 W(p_{2k})$ $\mathcal{O} \notin \partial_1 W(p_{2k})$. Moreover, given $B(p_{2k})$ a conj-equivariant 4-ball around p_{2k},*

$$\mathcal{H}_{2k} \cap B(p_{2k}) \approx (\partial_0 W(p_{2k}) \cap B(p_{2k}))_1$$

(2) *There exists a negative oval $\mathcal{O}$ such that the point $p_{2k+1} \in S^+_{2k+1}$ has the property: $\mathcal{O} \in \partial_0 W(p_{2k+1})$ $\mathcal{O} \notin \partial_1 W(p_{2k+1})$. Moreover, given $B(p_{2k+1})$ a conj-equivariant 4-ball around p_{2k+1},*

$$\mathcal{H}_{2k+1} \cap B(p_{2k+1}) \approx (\partial_0 W(p_{2k+1}) \cap B(p_{2k+1}))_1$$

(3) *There exists a negative oval $\mathcal{O}$ such that the point* $p_{2k+2} \in S^+_{2k+2}$ *has the property:* $\mathcal{O} \in \partial_0 W(p_{2k+2})$ $\mathcal{O} \notin \partial_1 W(p_{2k+2})$. *Moreover, given* $B(p_{2k+2})$ *a conj-equivariant 4-ball around* p_{2k+2},

$$\mathcal{H}_{2k+2} \cap B(p_{2k+2}) \approx (\partial_0 W(p_{2k+2}) \cap B(p_{2k+2}))_1$$

DEFINITION 2.10. We shall call 3-*point of the second kind* a 3-point verifying all assumptions of Lemma 2.9 of Chapter 4.

proof

Our proof is based on arguments similar to the one given in the proof of Lemma 2.7 of Chapter 4.

We shall use local description of Harnack curves $\mathcal{H}_{m+j}$ $j \geq 0$ inside $U(p)$ provided by the patchworking theory.

Recall that for any $j \geq 0$, there exists an homeomorphism $\tilde{h} : \mathbf{C}\mathcal{H}_{m+j} \cap U(p) \to \{(x,y) \in (\mathbf{C}^*)^2 | b^S_{m+j} = 0\} \cap U(p)$ such that $\tilde{h}(a) = (x_0, y_0)$ is a critical point of b^S_{m+j}.

(1) Let $\mathcal{H}_{2k}$ be the Harnack curve of degree $2k$, $(k > 3)$ and B_{2k} be Harnack polynomial of degree $2k$.

From proposition 3.6 of Chapter 2, a point p_{2k} of S^+_{2k} is associated with a. Set $b_{2k}(p_{2k}) = c_0$. Then, when $c = c_0$, the non-empty positive oval touches a negative (inner) oval; as c increases from c_0 to $c_0 + \epsilon$, one negative oval $\mathcal{O}^-$ disappears. (In the pencil of curves, $\mathcal{O}^-$ as oval of $\mathcal{H}_{2k-1}$ has been created in the construction of $\mathcal{H}_{2k-1}$ from $\mathcal{H}_{2k-2} \cup L$).

(2)

Let $\mathcal{H}_{2k+1}$ be the Harnack curve of degree $2k+1$ and B_{2k+1} be a Harnack polynomial of degree $2k+1$.

Consider the convex polygon $K \subset T_{2k+1}$ with vertices $(c,d), (c,d+2), (c+1, d+2), (c+2,d), (c+2,d+1)$. It is contained in triangle T_{2k+1} and triangulated by the triangulation τ of T_{2k+1}.

Let $U(K^0)$ be the subset $\rho^{2k+1}(\mathbf{R}_+ K^0 \times U^2_{\mathbf{C}})$ of $\mathbf{C}P^2$. Obviously, $U(p) \subset U(K^0)$.

Denote b^K_{2k+1} the truncation of b_{2k+1} on the monomials $x^c y^d$, $x^c y^{d+1}$, $x^c y^{d+2}$, $x^{c+1} y^d$, $x^{c+1} y^{d+1}$, $x^{c+1} y^{d+2}$, $x^{c+2} y^d$, $x^{c+2} y^{d+1}$.

According to the patchworking theory, the truncation b^K_{2k+1} of b_{2k+1} is ϵ-sufficient for b_{2k+1} in $U(K^0)$.

Therefore, $\tilde{h} : \mathbf{C}\mathcal{H}_{2k+1} \cap U(p) \to \{(x,y) \in (\mathbf{C}^*)^2 | b^S_{2k+1} = 0\} \cap U(p)$ extends to the homeomorphism $\tilde{h} : \mathbf{C}\mathcal{H}_{2k+1} \cap U(K^0) \to \{(x,y) \in (\mathbf{C}^*)^2 | b^K_{2k+1} = 0\} \cap U(K^0)$ such that $\tilde{h}(a) = (x_0, y_0)$ $x_0.y_0 < 0$. According to proposition 3.1 of Chapter 2, one can assume that $(x_0, -y_0)$ is a critical point of $b^S_{2k+1} = b^S_{2k}$. It is easy to see that, up to modify coefficients of the polynomials b^K_{2k+1} (without changing the order and the topological structure of b_{2k+1}), the point $(x_0, -y_0)$ $x_0 > 0, -y_0 > 0$ is a critical point of b^K_{2k+1} with $b_{2k+1}(x_0, -y_0) > 0$.

Let d_{2k+1} be the unique polynomial such that $b_{2k+1} = b^K_{2k+1} + d_{2k+1}$. Consider curves of the Petrovskii's pencil over $\mathcal{H}_{2k+1}$ outside $d_{2k+1} = 0$, as level curves of the function $\frac{b^K_{2k+1} - c}{d_{2k+1}}$.

Inside $(b^K_{2k+1} - c = 0) \backslash (d_{2k+1} = 0)$, these curves have critical points the singular points of $b_{2k+1} - c = 0$.

Bringing together Petrovskii's theory and the implicit function theorem applied to one-parameter polynomial $x_0^{2k+1}.(b_{2k+1} - c)$ with parameter c, it follows that a point p_{2k+1} with $b_{2k+1}(p_{2k+1}) = c_0 > 0$ is associated with a.

When $c = c_0$, the one-side component of the curve touches itself. Then, as c increases from c_0 to $c_0 + \epsilon$ one negative oval $\mathcal{O}^-$ disappears. (In the pencil of curves, $\mathcal{O}^-$ as oval of $\mathcal{H}_{2k-1}$ has been created in the construction of $\mathcal{H}_{2k-1}$ from $\mathcal{H}_{2k-2} \cup L$).

(3) Let $\mathcal{H}_{2k+2}$ be the Harnack curve of degree $2k+2$ and B_{2k+2} be the Harnack polynomial of degree $2k+2$

Consider J the square with vertices $(c,d), (c,d+2), (c+2,d), (c+2,d+2)$. It is contained in the triangle T_{2k+2} and triangulated by the triangulation τ of T_{2k+2}.

Let $U(J^0)$ be the subset $\rho^{2k+2}(\mathbf{R}_+ J^0 \times U^2_{\mathbf{C}})$ of $\mathbf{C}P^2$. Obviously, $U(p) \subset U(K^0) \subset U(J^0)$.

Denote b^J_{2k+2} the truncation of b_{2k+2} on the monomials $x^c y^d$, $x^c y^{d+1}$, $x^c y^{d+2}$, $x^{c+1} y^d$, $x^{c+1} y^{d+1}$, $x^{c+1} y^{d+2}$, $x^{c+2} y^d$, $x^{c+2} y^{d+1}$,$x^{c+2} y^{d+2}$.

According to patchworking theory, the truncation b^J_{2k+2} of b_{2k+2} is ϵ-sufficient for b_{2k+2} in $U(J^0)$.

Therefore,
$\tilde{h} : \mathbf{C}\mathcal{H}_{2k+2} \cap U(p) \to \{(x,y) \in (\mathbf{C}^*)^2 | b^S_{2k+2} = 0\} \cap U(p)$ extends to the homeomorphism $\tilde{h} : \mathbf{C}\mathcal{H}_{2k+2} \cap U(J^0) \to \{(x,y) \in (\mathbf{C}^*)^2 | b^J_{2k+2} = 0\} \cap U(J^0)$ such that $\tilde{h}(a) = (x_0, y_0)$ $x_0.y_0 < 0$.

It is easy to see that up to modify coefficients of the polynomials b^J_{2k+2} (without changing the order and the topological structure of b_{2k+2}), the point $(x_0, -y_0)$ $x_0 > 0, y_0 < 0$ is a critical point of b^J_{2k+2} with $b^J_{2k+2}(x_0, -y_0) > 0$.

According to the Petrovskii's theory, from an argumentation similar to the previous one, it follows that a point p_{2k+2} with $b_{2k+2}(p_{2k+2}) = c_0 > 0$ is associated with a.

When $c = c_0$, the negative oval $\mathcal{O}^-$ touches another negative oval. (In the pencil of curves, this last negative oval as oval of $\mathcal{H}_{2k+1}$ is, in the patchworking scheme, situated in front of $\mathcal{O}^-$ and has been created in the construction of $\mathcal{H}_{2k+1}$ from $\mathcal{H}_{2k} \cup L$).

Then, as c increases from c_0 to $c_0 + \epsilon$, one negative oval disappears.

Let $j \geq 3$ be an integer and $\mathcal{H}_{2k+j}$ be the Harnack curve of degree $2k+j$.

Let B_{2k+j} be a Harnack polynomial of degree $m+j$. As previously, consider b^J_{2k+j} the truncation of b_{2k+j} on the monomials $x^c y^d$, $x^c y^{d+1}$, $x^{c+1} y^d$, $x^{c+1} y^{d+1}$. According to the patchworking theory, the truncation b^K_{2k+j} of b_{2k+j} is ϵ-sufficient for b_{2k+j} in $U(J^0)$. Therefore, $\tilde{h} : \mathbf{C}\mathcal{H}_{2k+j} \cap U(p) \to \{(x,y) \in (\mathbf{C}^*)^2 | b^S_{2k+j} = 0\} \cap U(p)$. extends to the homeomorphism $\tilde{h} : \mathbf{C}\mathcal{H}_{2k+j} \cap U(K^0) \to \{(x,y) \in (\mathbf{C}^*)^2 | b^K_{2k+1} = 0\} \cap U(K^0)$ such that $\tilde{h}(a) = (x_0, y_0)$ $x_0.y_0 < 0$.

From an argumentation similar to the previous one, it follows that a point p_{2k+j} of S^+_{2k+j} equivalent to p_{2k+2} is associated with a.

Therefore, a *3-point* is associated with a. Q.E.D

Remark 2.11. Given $\mathcal{H}_{2k}$ the Harnack of degree $2k$. It follows immediately from the Lemmas 2.6 of Chapter 4, 2.7 of Chapter 4, 2.9 of Chapter 4 above that $(k-2)^2$ 3-points are associated to crossing of curves $\mathcal{H}_{2k-3-j} \cup L$ ($L \approx \mathbf{C}l_{2k-3-j}$) $(0 \leq j \leq (2k-7))$.

Likewise, given $\mathcal{H}_{2k+1}$ the Harnack curve of odd degree $2k+1$, $(k-2)^2$ 3-points are associated to crossing of curves $\mathcal{H}_{2k-3-j} \cup L$ ($L \approx \mathbf{C}l_{2k-3-j}$) $(0 \leq j \leq (2k-7))$, and $(2k-3)$ 2-points are associated with crossing points of $\mathcal{H}_{2k-1} \cup L$. Hence, for any Harnack curve $\mathcal{H}_m$ of degree m with polynomial B_m, one can recover from the Lemmas above local topological meaning of critical points of index 1 of B_m.

CHAPTER 5

Perestroika theory on Harnack curves

-Construction of any curve of degree m with non-empty real part-

In this chapter, we shall give a method of construction of any curve of a given degree m which provides description of pairs $(\mathbf{C}P^2, \mathbf{C}\mathcal{A}_m)$ up to conj-equivariant isotopy. In a few words, our method of construction of curves of degree m is based on the definition of a chain of modification on the real connected components of the Harnack curve $\mathcal{H}_m$.

We shall divide this section into three sections. Our method of construction of algebraic curve uses invariants of real algebraic curves similar to the Arnold's invariants of generic immersion of the circle into the plane. These invariants and their analogous for algebraic curves were respectively introduced in [**4**] and [**19**]. In the first section, we state the problem of construction of algebraic curves from the pair $(\mathbf{C}P^2, \mathbf{C}\mathcal{H}_m)$ and recall definitions of these invariants. Classification of pairs $(\mathbf{C}P^2, \mathbf{C}\mathcal{A}_m)$ leads us to distinguish curves with orientable set of real points usually called curves of type I, and curves with non-orientable set of real points usually called curves of type II. In the second section, we state a method of construction of curves of type I. Then, in the third section, we enlarge the method to curves of type II. The main result of this section is gathered in Theorem 2.9 of Chapter 5 and Theorem 3.7 of Chapter 5. This statement (Theorem 2.9 of Chapter 5 and 3.7 of Chapter 5) is the counterpart of Theorem 3.9 of Chapter 2 established for Harnack curves. Namely, we present any curve $\mathcal{A}_m$ of degree m with non-empty real part up to conj-equivariant isotopy of $\mathbf{C}P^2$ as follows: Outside a finite number of 4-balls $B(a_i)$ globally invariant by complex conjugation, $\mathcal{A}_m$ is the union of m non-intersecting projective lines; inside any 4-ball $B(a_i)$ it is the perturbation of a crossing.

1. Introduction-Classical Topological Facts-

In any cases, (curves $\mathcal{A}_m$ with an arbitrary number of real components and arbitrary type), we shall bring the problem of description of $(\mathbf{C}P^2, \mathbf{C}\mathcal{A}_m)$ to the problem of description of all possible moves of the real components of $\mathcal{H}_m$. Let us recall in introduction general facts (see [**18**] for example) which will state the space of moves providing an arbitrary curve of degree m from the Harnack curve $\mathcal{H}_m$.

Introduction. Passing from polynomials to real set of points, it follows easily that real algebraic curves form a real projective space of dimension $\frac{m(m+3)}{2}$. We shall denote this space by the symbol $\mathbf{R}\mathcal{C}_m$. It easy to see that one can trace the real part of a curve of degree m through any $\frac{m(m+3)}{2}$ real points, and that it is uniquely defined for points in general position. Moreover, real algebraic curves can

be considered as complex curves of special kind. Similarly, passing from polynomials to complex set of points, it follows easily that complex algebraic curves of degree m form a complex projective space of dimension $\frac{m(m+3)}{2}$. We shall denote this space by $\mathbf{C}\mathcal{C}_m$. Obviously, $\mathbf{R}\mathcal{C}_m$ coincides with the real part of $\mathbf{C}\mathcal{C}_m$. It is easy to see that through any $\frac{m(m+3)}{2}$ points one can always trace a curve of degree m. Moreover, the set of $\frac{m(m+3)}{2}$-points for which such a curve is unique is open and dense in the space of all $\frac{m(m+3)}{2}$-points subset of $\mathbf{C}P^2$.

Let $\mathbf{R}\mathcal{D}_m$ denote the subset of $\mathbf{R}\mathcal{C}_m$ corresponding to real singular curves. Let $\mathbf{C}\mathcal{D}_m$ denote the subset of $\mathbf{C}\mathcal{C}_m$ corresponding to singular curves.

We call a path in the complement $\mathbf{R}\mathcal{C}_m \backslash \mathbf{R}\mathcal{D}_m$ of the discriminant hypersurface in $\mathbf{R}\mathcal{C}_m$ a *rigid isotopy* of real points set of nonsingular curves of degree m. We call a smooth path in the complement $\mathbf{C}\mathcal{C}_m \backslash \mathbf{C}\mathcal{D}_m$ of the discriminant hypersurface in $\mathbf{C}\mathcal{C}_m$ a *rigid isotopy* of complex point set of nonsingular curves of degree m.

These definitions give rise naturally to the classification problem of non-singular curves of degree m up to rigid isotopy.

From the complex viewpoint, the rigid isotopy classification problem has trivial solution: the complex point sets of any two nonsingular curves of degree m are rigidly isotopic. Moreover, they are diffeotopic in $\mathbf{C}P^2$. From the real viewpoint, even if we consider only rigid isotopy, this property can not be extended.

State of the problem. From now on, one can state some properties of the sets of points $\mathbf{C}\mathcal{A}_m$, $\mathbf{R}\mathcal{A}_m$ of a curve $\mathcal{A}_m$. Let $\mathcal{H}_m$ be the Harnack curve of degree m and $\mathcal{A}_m$ be a curve of degree m such that the pair $(\mathbf{R}P^2, \mathbf{R}\mathcal{A}_m)$ is non-homeomorphic to the pair $(\mathbf{R}P^2, \mathbf{R}\mathcal{H}_m)$. From the elementary topological facts recalled previously, there exists a smooth path $h : [0,1] \to \mathbf{C}\mathcal{C}_m \backslash \mathbf{C}\mathcal{D}_m$ with $h(0) = \mathbf{C}\mathcal{H}_m, h(1) = \mathbf{C}\mathcal{A}_m$ such that the family $\mathbf{C}h(t)$ is a diffeotopy of submanifolds of $\mathbf{C}P^2$. Thus, if there exists $B \subset \mathbf{C}P^2$ globally invariant by complex conjugation such that $h_t(\mathbf{C}\mathcal{H}_m) \cap B) \subset B$, $t \in [0,1]$, topological spaces $(B \cap \mathcal{H}_m)$ and $(B \cap \mathcal{A}_m)$ are isotopic in B.

According to corollary 3.5 of Chapter 2, assume $\mathcal{H}_m$ obtained via T-inductive construction of Harnack curves. From the Lemma 2.6 of Chapter 4, Lemma 2.7 of Chapter 4 and Lemma 2.9 of Chapter 4 of the previous section, any critical point (x_0, y_0) of index 1 of B_m is associated with a crossing a of one curve $\mathcal{H}_{m-i} \cup L$. Namely, there exists $U(p) = \rho^m(D(p,\epsilon) \times U^2_{\mathbf{C}}) \subset (\mathbf{C}^*)^2$ ϵ-neighborhood of $\mathbf{C}\mathcal{H}_m$ defined the interior Γ^0 of a face Γ of the triangulation of T_m such that :

(1) $a, (x_0, y_0) \in U(p)$
(2) the perturbation on the real part of $\mathcal{H}_m$ involved in the local topological meaning of (x_0, y_0) is a deformation on $\mathbf{C}\mathcal{H}_m \cap U(p)$

Let us denote by $\mathcal{P}_m$ the set of points p of T_m with the property that for any $p \in \mathcal{P}_m$, the ϵ-neighborhood $U(p)$ verifies the two properties above. Note that for any two distinct points p, p' the subsets $U(p)$ and $U(p')$ have empty-intersection.

It easy to see that the set $\mathcal{P}_m$ contains $\frac{m(m-1)}{2}$ points of T_m.
Moreover, for any $p \in \mathcal{P}_m$, $U(p) = \rho^m(D(p,\epsilon) \times U^2_{\mathbf{C}})$ contains conj-equivariant 4-balls $B(x_0, y_0)$ and $B(a)$ around (x_0, y_0) and a such that
$\mathbf{C}\mathcal{H}_m \cap B((x_0, y_0)) \neq \emptyset$,$\mathbf{C}\mathcal{H}_m \cap B(a) \neq \emptyset$.

Thus, according to the density of $\frac{m(m+3)}{2}$-uple of points which define uniquely a curve in the space of $\frac{m(m+3)}{2}$-uple of points of $\mathbf{C}P^2$, if there exists a diffeotopy of submanifolds of $\mathbf{C}P^2$ $h : [0,1] \to \mathbf{C}\mathcal{C}_m \backslash \mathbf{C}\mathcal{D}_m$ with $h(0) = \mathbf{C}\mathcal{H}_m, h(1) = \mathbf{C}\mathcal{A}_m$ such that $h_t(\mathbf{C}\mathcal{H}_m \cap U(p)) \subset U(p)$, $t \in [0,1]$, the curve $\mathcal{A}_m$ is entirely defined from its restriction on the union $\cup_{\{p \in \mathcal{P}_m\}} U(p)$ of ϵ-neighborhood $U(p)$ taken over the $\frac{m(m-1)}{2}$ points of $\mathcal{P}_m$.

In such a way, we bring the problem of construction of a curve $\mathcal{A}_m$ to the definition of a path $h : [0,1] \to \mathbf{C}\mathcal{C}_m \backslash \mathbf{C}\mathcal{D}_m$ with $h(0) = \mathbf{C}\mathcal{H}_m, h(1) = \mathbf{C}\mathcal{A}_m$.

Varieties of irreducible curves of type I, degree m, and genus g. As already introduced for smooth curves, curves with orientable real set of points are called curves of type I.

In what follows, we shall consider smooth curves of type I in the broader set of irreducible curves of type I. Any irreducible curve of type I is such that the real part of its normalization divides the set of complex points of its normalization into two halves. We shall call the images of the halves of the normalization in the set of points of the curve, the *halves* of the curve. Each of the halves is oriented and induces an orientation on the real part as on its boundary. We shall consider irreducible curves of degree m, genus g and type I with a distinguished complex orientation. Curves of this kind constitute a finite-dimensional stratified real algebraic variety we shall denote $\mathcal{C}_{I;g}$.

We shall set $\mathcal{C}_{I,m} = \cup_{0 \leq g \leq \frac{(m-1)(m-2)}{2}} \mathcal{C}_{I;g}$ the subset of $\mathcal{C}_m$ constituted by all irreducible curves of degree m, genus g, $0 \leq g \leq \frac{(m-1)(m-2)}{2}$, and type I.

Passing from polynomials to real (resp, complex) set of points, it follows easily that varieties $\mathcal{C}_{I;g}$ define subspaces $\mathbf{R}\mathcal{C}_{I;g}$ (resp,$\mathbf{C}\mathcal{C}_{I;g}$) of the space $\mathbf{R}\mathcal{C}_m$ (resp, $\mathbf{C}\mathcal{C}_m$.)

We shall set $\mathbf{R}\mathcal{C}_{I,m} = \cup_{0 \leq g \leq \frac{(m-1)(m-2)}{2}} \mathbf{R}\mathcal{C}_{I;g}$ the subset of $\mathbf{R}\mathcal{C}_m$ constituted by real set of points of irreducible curves of degree m and type I, $\mathbf{C}\mathcal{C}_{I,m} = \cup_{0 \leq g \leq \frac{(m-1)(m-2)}{2}} \mathbf{C}\mathcal{C}_{I;g}$ the subset of $\mathbf{C}\mathcal{C}_m$ constituted by complex set of points of irreducible curves of degree m and type I. Obviously, $\mathbf{R}\mathcal{C}_{I,m}$ coincides with the real part of $\mathbf{C}\mathcal{C}_{I,m}$.

Let us study the set of singular algebraic curves in the variety $\mathcal{C}_{I;g}$ of curves of degree m, genus g and type I.

Generic curves

Following usual terminology, we call *ordinary double point singularity* of a real irreducible algebraic curve $\mathcal{A}_m$ a non-degenerate singular point of $\mathcal{A}_m$. All ordinary double points singularity are equivalent from the complex viewpoint. From the real viewpoint, one distinguishes several types of such points.

(1) real double point of intersection of two real branches called crossing
(2) real point of intersection of two complex branches conjugated to each other called *solitary double point*
(3) imaginary double point of intersection of the different halves of $\mathbf{C}\mathcal{A}_m \backslash \mathbf{R}\mathcal{A}_m$ denoted α-point.
(4) imaginary double point of self-intersection of one of the halves of $\mathbf{C}\mathcal{A}_m \backslash \mathbf{R}\mathcal{A}_m$ denoted β-point.

Define a *generic curve* as a real irreducible algebraic curve with only ordinary double singularities. As it is well known, generic curves in the variety $\mathcal{C}_{I;g}$ constitute a Zarisky open set in the variety $\mathcal{C}_{I;g}$.

For a generic curve $\mathcal{A}_m$ of degree m and genus g ,$0 < g < \frac{(m-1)(m-2)}{2}$, by *smoothing* of its real point set $\mathbf{R}\mathcal{A}_m$ we shall understand a smooth oriented 1-dimensional submanifold of $\mathbf{R}P^2$ obtained from $\mathbf{R}\mathcal{A}_m$ by the modification at each double point determined by the complex orientation (see figure 5.1).

By smoothing of its complex point set we shall understand a smooth oriented 1-dimensional complex submanifold of $\mathbf{C}P^2$ obtained from $\mathbf{C}\mathcal{A}_m$ by the modification at each real double point as above and by the modification at each complex double point. The two complex branches which merge in α-point (resp β-point) become after smoothing two branches of different halves of $\mathbf{C}\mathcal{A}_m \backslash \mathbf{R}\mathcal{A}_m$ (resp, two branches of one of the halves of $\mathbf{C}\mathcal{A}_m \backslash \mathbf{R}\mathcal{A}_m$)

We shall call *smoothing* of a generic curve $\mathcal{A}_m$, the smooth 1-dimensional complex submanifold of $\mathbf{C}P^2$ deduced from $\mathcal{A}_m$ by smoothing of $\mathbf{R}\mathcal{A}_m$ and $\mathbf{C}\mathcal{A}_m$ as described above. We shall say that the singular points of $\mathcal{A}_m$ are smoothed.

Discriminant Hypersurface

Consider now the complement of the set of generic curves of all real algebraic curves in the variety $\mathcal{C}_{I;g}$. It can be considered as a discriminant hypersurface of which strata consist of curves with only one singular point which is not an ordinary double point. One can distinguish six main strata defined by the type of the singularity of the curves it contains:

(1) real cusp,
(2) real point of direct ordinary tangency,
(3) real point of inverse ordinary tangency,
(4) real point of ordinary tangency of two imaginary branches,
(5) real ordinary triple point of intersection of three real branches,
(6) real ordinary triple point of intersection of a real branch and two conjugate imaginary branches.

Perestroika

A *generic path* in the variety $\mathcal{C}_{I;g}$ intersects the discriminant hypersurface in a finite number of points, and these points belong to the main strata.

We call *perestroika* a change experienced by a generic path in the space $\mathcal{C}_{I;g}$ when it goes through the main strata. The notion of perestroika was initially introduced in the context of generic immersion of the circle into the plane (see for example [**4**], [**19**]).

Given a perestroika, we call *smoothed perestroika* the change obtained by smoothing the fragments involved in the perestroika.

Moreover, to define how a generic path crosses the strata of the discriminant hypersurface one has to specify a coorientation of the strata. Let us define the *positive direction* of a perestroika. The opposite direction is naturally called *negative direction*.

(1) In the case of cusp, define the *positive direction* the direction from curve with one more crossing point to curve with one more solitary double point.
(2) In case of real point of tangency, there is a natural *positive direction* from curve with less real double points to curve with more double points.
(3) In case of triple-point, the coorientation of the stratum can be defined as follows. The crossing of a triple point by a path gives rise to the *vanishing triangle*: the dying triangle which existed just before the crossing and the new born existed just after it. A cyclic order on the sides of the triangle is given by the order the sides are visited. In the case of triple-point, the coorientation rule assigns *signs* to triangle. Let q be the number of the sides of the vanishing triangle whose directions coincide with that given by the cyclical order. The sign of a triangle is $(-1)^q$. The crossing of the triple point is *positive* (resp, *negative*) if the newborn triangle is positive (resp, *negative*) (and hence the dying one is negative (resp, positive)).

Let us study perestroikas and smoothed perestroikas in the positive direction. Properties of perestroikas in the negative direction can be naturally deduced from properties in the positive one.

We shall first consider perestroikas in which no imaginary double point is involved; namely, cusp perestroika and triple-point perestroika. It is obvious that the corresponding smoothed perestroikas do not change the complex part. Therefore, we shall consider only real part.

(1) Cusp Perestroika
It is easy to see that smoothed cusp perestroika does not change real part.
(2) Triple-point perestroika.

Relatives orientation of the three real branches of a real triple-point gives rise to distinguish two kinds of real triple-point. Consider vectors at the triple point tangent to the branches and directed according to their orientations. If one of the vectors can be presented as linear combination of the two others, then the triple-point is said *weak* (see figure 5.2); otherwise it is said *strong*(see figure 5.3) Smoothed weak triple-point perestroika does not change real part.

Smoothed strong triple-point perestroika changes real part as shown in figure 5.(3.b).

Then, consider perestroika in which imaginary singularities are involved.

Introduce a function J counting the difference between the number of (resp, smoothed) α-points and β-points under (resp, smoothed) perestroika. Obviously, if $J(\alpha)$ and $J(\beta)$ are the values of J under one perestroika π in the positive direction the function J takes values $-J(\alpha)$, $-J(\beta)$ under the perestroika π in the negative direction.

(1) As already known, relative orientation of the real branches of a real self-tangency point gives rise to distinguish real point of direct ordinary tangency and real point of inverse ordinary tangency.
    (a) It is easy to see that smoothed real direct ordinary tangency perestroika does not change smoothed real part. Nonetheless, the complex part is changed in such a way that:
    $J(\alpha) = 0, J(\beta) = -2$ under real direct ordinary tangency perestroika.

   (b) Smoothed real inverse tangency perestroika changes the real part (see figure 5.4). Besides, the complex part is changed in such a way that : $J(\alpha) = -2, J(\beta) = 0$ under real inverse tangency perestroika.
(2) Perestroika of solitary self tangency is as follows: two solitary double points come from the world of imaginary to form a solitary self-tangency point; then arise two solitary double points with opposite orientation. Besides, the complex part is changed in such a way that: $J(\alpha) = -2, J(\beta) = 0$ under a solitary inverse tangency perestroika.
(3) Perestroika of a triple-point with imaginary branches changes the real part as shown in figure 5.5. Besides, it changes the complex part in such a way that $J(\alpha) = +2, J(\beta) = -2$.

2. Construction of Curves of type I

In this section, we shall consider curves $\mathcal{A}_m$ of degree m and type I with pair $(\mathbf{R}P^2, \mathbf{R}\mathcal{A}_m)$ non-homeomorphic to the pair $(\mathbf{R}P^2, \mathbf{R}\mathcal{H}_m)$. As already noticed, for any curve $\mathcal{A}_m$ of type I, the pair $(\mathbf{R}P^2, \mathbf{R}\mathcal{A}_m)$ with orientation of the real point set $\mathbf{R}\mathcal{A}_m$ provides the pair $(\mathbf{C}P^2, \mathbf{C}\mathcal{A}_m)$, up to conj-equivariant isotopy of $\mathbf{C}P^2$. Bringing together the recursive Morse-Petrovskii's theory for Harnack's curves $\mathcal{H}_m$ of chapter 4 and properties of generic paths in the varieties $\mathcal{C}_{I;g}$, we define in Proposition 2.4 of Chapter 5 a path $S : [0, 1] \to \mathbf{R}\mathcal{C}_m$ with $S(0) = \mathbf{R}\mathcal{H}_m$, $S(1) = \mathbf{R}\mathcal{A}_m$. Such path is obtained from lifting a generic path in the space of generic immersion of the circle into $\mathbf{R}^2$ and is described as a set of moves defined on the connected real components of $\mathbf{R}\mathcal{H}_m$.
In this way, the classification of M-curves of a prescribed degree m amounts to the description of all the possible locations of the real components of the Harnack curve $\mathcal{H}_m$.

The first two parts of this section are devoted to the definition of the lifting. Along a generic path in the space of generic immersion of the circle into $\mathbf{R}^2$, three type of events "perestroikas" (namely, the instateneous triple crossings and the inverse and direct self-tangency) may happen. By means of perestroikas and their counterparts for algebraic curves, we bring the problem of the definition of the path S to the one of lifting perestroikas in the space $\mathbf{R}\mathcal{C}_m$.
In the third part, we give in Theorem 2.9 of Chapter 5 a description of pairs $(\mathbf{C}P^2, \mathbf{C}\mathcal{A}_m)$ up to conj-equivariant isotopy which extends the properties of the Harnack curve $\mathcal{H}_m$ stated in Theorem 3.9 of Chapter 2 to any smooth curve of type I.

Smoothing generic immersion of the circle into $\mathbf{R}^2$. Recall that by a generic immersion of the circle S^1 into the plane $\mathbf{R}^2$, one means an immersion with only ordinary double points of transversal intersection, namely without triple points and without points of self-tangency. The space of all immersions is an infinite-dimensional manifold which consists of an infinite countable set of irreducible components described by H.Whitney [**20**] in 1937.

Whitney's Theorem (1937) [**20**] *The space of the immersions of a circle into the plane with the same Whitney index is pathwise connected*

The *Whitney index* of an immersion of an oriented curve into the plane is the rotation number of the tangent vector (the degree of the Gauss map).

Arbitrary differentiable immersion of the circle into the plane does not admit complexification. Nonetheless, we shall generalize Whitney's Theorem to the case of real algebraic curves of type I.

Let us recall in introduction the following property of the genus.

Genus Property If a collection of disjoint circles embedded into a closed orientable surface of genus g does not divide the surface, the number of circles is at most g. In particular $g+1$ disjoint circles always divide the surface.
The set of complex points of any smooth curve of degree m is an orientable surface of genus $g_m = \frac{(m-1)(m-2)}{2}$. Therefore, one can easily conjecture the following one-to-one correspondence between the $g_m + 1$ (resp, $l \leq g_m + 1$) real connected components of an M-curve (resp, an $M-i$-curve) $\mathcal{A}_m$ and the g_m 1-handles and the sphere S^2: the sphere S^2 and each 1-handle contains one real connected component of $\mathbf{R}\mathcal{A}_m$ which divides it into two halves (resp, the sphere S^2 and each 1-handle contains a part of a real connected component of $\mathbf{R}\mathcal{A}_m$; such part divides it into two halves.)

Call *regular curve*, the smooth oriented submanifold of $\mathbf{R}P^2$ deduced from the generic immersion of the circle by modification at each real double point which is either the Morse modification in $\mathbf{R}^2$ in the direction coherent to a complex orientation or the *Morse modification at infinity in* $\mathbf{R}P^2$ (in the direction coherent to a complex orientation) of the double point which associates to the double point of $\mathbf{R}^2$ two points of the line at infinity of $\mathbf{R}P^2$.

In Lemma 2.1 of Chapter 5, we give a generalization of Whitney's theorem to the case of real algebraic curves of type I.

LEMMA 2.1. *Let $\mathcal{A}_m$ be a smooth curve of degree m and type I, then its real point set $\mathbf{R}\mathcal{A}_m \subset \mathbf{R}P^2$ is a regular curve deduced from a generic immersion of the circle S^1 into the plane $\mathbf{R}^2$ with $\frac{(m-1)(m-2)}{2} \leq n \leq \frac{(m-1)(m-2)}{2} + [\frac{m}{2}]$, double points and Whitney index $\frac{(m-1)(m-2)}{2} + 1$. The smoothing is such that $\frac{(m-1)(m-2)}{2}$ double points are smoothed in $\mathbf{R}^2$ by Morse modification in the direction coherent to a complex orientation. The $n - \frac{(m-1)(m-2)}{2}$ others double points are smoothed by Morse modification at infinity in $\mathbf{R}P^2$.*

REMARK 2.2. One can associate to any smooth curve of type I a three-dimensional rooted tree.
Recall [**4**] that a generic immersion of the circle into the plane is a *tree-like curve* if any of its double points subdivides it into two disjoints loops. On the assumption of Lemma 2.1 of Chapter 5, let $\mathcal{A}_m$ be a smooth curve of degree m and type I, and ϕ the corresponding generic immersion of the circle S^1 into the plane $\mathbf{R}^2$ with $\frac{(m-1)(m-2)}{2} \leq n \leq \frac{(m-1)(m-2)}{2} + [\frac{m}{2}]$. Smooth the double-points of ϕ which result from points of $\mathbf{R}\mathcal{A}_m$ at infinity in such a way that it results a curve in $\mathbf{R}^3$ with the property that any of its double points subdivides it into two disjoints loops; it follows the rooted-tree associated to $\mathcal{A}_m$.
In case of Harnack curves $\mathcal{H}_m$, an immediate definition of the root of its rooted-tree is the following: assume $\mathcal{H}_m$ obtained via patchworking process;
for odd m, the root consists of the part of the odd component of the curve which intersects T_m;

for even m, the root consists of the part of the positive oval (non-empty for $m \geq 4$) which intersects T_m.

proof:

Our proof makes use of properties of the complex point set $\mathbf{C}\mathcal{A}_m$ embedded in $\mathbf{C}P^2$.

Consider the usual handlebody decomposition of $\mathbf{C}P^2 = B_0 \cup B_1 \cup B$ where B_0,B_1,B are respectively 0, 2 and 4 handles.

The balls B_0 and B_1 meet along an unknotted solid torus $S^1 \times B^2$. The gluing diffeomorphism $S^1 \times B^2 \to S^1 \times B^2$ is given by the $+1$ framing map. In such a way, the canonical $\mathbf{R}P^2$ can be seen as the union of a Möbius band $\mathcal{M}$ and the disc $D^2 \subset B$ glued along their boundary. The Möbius band $\mathcal{M}$ lies in $B_0 \cap B_1 \approx S^1 \times D^2$ with $\partial\mathcal{M}$ as the $(2,1)$ torus knot, and $D^2 \subset B$ as the properly imbedded unknotted disc. The complex conjugation switches B_0 and B_1 and lets fix $\mathcal{M}$, it rotates B around D^2.

The set of complex points of $\mathcal{A}_m$ is an orientable surface of genus $g_m = \frac{(m-1)(m-2)}{2}$, i.e it is diffeomorphic to a sphere S^2 with g_m 1-handles $S^2_{g_m}$. We shall denote h the diffeomorphism $h : \mathbf{C}\mathcal{A}_m \to S^2_{g_m} \subset \mathbf{R}^3$.

Recall that surfaces $S^2_{g_m}$ are constructed as follows. From the sphere S^2, $(m-1)$ pairwise non-intersecting open discs are removed, and then the resulting holes are closed up by g_m orientable cylinders $\approx S^1 \times D^1$ connecting the boundary circles of the discs removed.

Assume S^2 provided with a complex conjugation with fixed point set a circle S^1 which divides the sphere S^2 into two halves. Without loss of generality, one can assume that any disc removed from S^2 intersects S^1 and the two halves of S^2. In such a way, $\mathbf{R}\mathcal{A}_m \subset \mathbf{R}P^2$ intersects each 1-handle.

Fix D^2 the two disc of $\mathbf{R}P^2 = \mathcal{M} \cup D^2$ in such a way that the boundary circle of D^2 is S^1 and therefore each one handle belongs to the solid torus $S^1 \times D^2 \supset \mathcal{M}$.

Let $D^2_\epsilon \supset D^2$ be the disc D^2 thickened. Since the interior $(D^2_\epsilon)^0$ of D^2_ϵ is homeomorphic to $\mathbf{R}^2$, one can project $\mathbf{R}\mathcal{A}_m$ (up to homeomorphism $\mathbf{R}^2 \approx (D^2_\epsilon)^0$) in a direction perpendicular to $\mathbf{R}^2$ onto $\mathbf{R}^2$. We may suppose that the direction of the projection is generic i.e all points of self-intersection of the image on $\mathbf{R}^2$ are double and the angles of intersection are non-zero.

Let $r : \mathbf{R}^3 \to \mathbf{R}^2$ be the projection which maps $h(\mathbf{R}\mathcal{A}_m) \subset \mathbf{R}^3$ to $\mathbf{R}^2$.

Consider an oriented tubular fibration $N \to \mathbf{C}\mathcal{A}_m$. Since $\mathbf{C}\mathcal{A}_m$ is diffeomorphic to a sphere S^2 with g_m 1-handles, one can consider the restriction of $\mathbf{C}\mathcal{A}_m$ diffeomorphic to each torus T^2 given by the sphere S^2 with one of the g_m 1-handles. The oriented tubular neighborhood of $h(\mathbf{C}\mathcal{A}_m) \cap T^2$, as oriented tubular neighborhood of the torus T^2, intersects the solid torus $S^1 \times D^2 \supset \mathcal{M}$ with $\partial M \subset S^1 \times S^1$ as the $(2,1)$ torus knot. Hence, since in $\mathbf{C}P^2$ each real line is split by its real part into two halves lines conjugate to each others, and 2 disjoint circles always divide the torus; the real part of the restriction of $h(\mathbf{C}\mathcal{A}_m) \cap T^2$ belongs to the boundary of the Möbius band in such a way that its projection to $\mathbf{R}^2$ gives one crossing. Besides these doubles, some double-points of $r(h(\mathbf{R}\mathcal{A}_m))$ may result either from two points of $\mathbf{R}\mathcal{A}_m$ which belong to two different handles or from two points which belong respectively to a 1-handle and to S^2. It is easy to deduce from the relative location of the sphere S^2 and the 1-handles that the projection of $\mathbf{R}\mathcal{A}_m$ leads to

an even number of these double-points of $r(h(\mathbf{R}\mathcal{A}_m))$. Therefore, the projection $r(h(\mathbf{R}\mathcal{A}_m))$ contains $g_m(mod2)$ crossings and $\mathbf{R}\mathcal{A}_m$ can be seen as a regular curve.

By use of the Whitney expression of the index ind of a plane curve with n double points - $ind = \sum \epsilon_i \pm 1$ where the summation is over the set of n double points, ϵ_i is a sign associated to each double point, and the term $\pm$ depends on the orientations of the curve and of the plane - we shall deduce the Whitney index of $r(h(\mathbf{R}\mathcal{A}_m))$.

According to the definition of $\mathbf{R}P^2$, each real part of $h(\mathbf{R}\mathcal{A}_m)$ contained in a 1-handle has the same projection to $\mathbf{R}^2$ as the $(2,1)$ torus knot on the torus defined from the sphere S^2 with the 1-handle.

Therefore, according to the gluing diffeomorphism
$S^1 \times B^2 \to S^1 \times B^2$, given an orientation on the circle S^1 induced by one of the halves of S^2, the projection of each real part of $h(\mathbf{R}\mathcal{A}_m)$ contained in a 1-handle gives a crossing with sign $+1$ of the plane $\mathbf{R}^2$. Besides, any 2 crossings which result from the projection of points of two different handles or of points of a 1-handle and the sphere S^2 have opposite Whitney index. Hence, $r(h(\mathbf{R}\mathcal{A}_m))$ has Whitney index $g_m + 1$.

Moreover, since $\mathcal{A}_m$ is of type I, the real part $\mathbf{R}\mathcal{A}_m$ is deduced from $r(h(\mathbf{R}\mathcal{A}_m))$ by smoothings which are either standard Morse modifications in $\mathbf{R}^2$ at the double point determined by the complex orientation, or smoothings at infinity in $\mathbf{R}P^2\backslash\mathbf{R}^2$ which associate to a crossing its two pre-image under r^{-1}.

Two real branches of $r(h(\mathbf{R}\mathcal{A}_m))$ which intersect in a crossing become after smoothing two oriented non-intersecting real branches which belong either to two different connected components of $\mathbf{R}\mathcal{A}_m$ or to one branch which lies as a part of the $(2,1)$ torus knot on the boundary Möbius $\mathcal{M} \subset \mathbf{R}P^2$. According to the Rokhlin's orientation formula, for any $m > 2$, $\mathbf{R}\mathcal{A}_m$ has at least two connected components.

Therefore, for $m > 2$, g_m crossings are smoothed in $\mathbf{R}^2$ by Morse Modification in the direction coherent to the complex orientation and become two oriented non-intersecting real branches of distinct connected components of $\mathbf{R}\mathcal{A}_m$ in $\mathbf{R}P^2$. The other double-points are smoothed in such a way that it results two points of $\mathbf{R}\mathcal{A}_m$ in $\mathbf{R}P^2\backslash\mathbf{R}^2$.
According to the Bezout's theorem, since each of these $n - g_m$ double-points results from two points which belong to $\mathbf{R}P^2\backslash\mathbf{R}^2$ and the number of points in the intersection of $\mathbf{R}\mathcal{A}_m$ with the real line $\mathbf{R}P^2\backslash\mathbf{R}^2$ is less or equal to m and congruent to $m(mod2)$, the difference $n - g_m$ is less or equal to $[\frac{m}{2}]$. It is obvious that one can assume, without loss of generality, that these $[\frac{m}{2}]$ double points results from points of $\mathbf{R}\mathcal{A}_m$ which belong to two different handles or from two points which belong respectively to a 1-handle and to S^2.

Q.E.D.

Given a generic immersion of the circle S^1 into the plane $\mathbf{R}^2$ and $\mathcal{S}$ the set of its singular points. We shall call *partially regular curve* the smooth oriented submanifold of $\mathbf{R}P^2$ deduced from a generic immersion of the circle by modification of a set $\mathcal{K} \subset \mathcal{S}$, $\mathcal{K} \neq \mathcal{S}$, of real double points where modification at each real double point of $\mathcal{K}$ is either the Morse modification in $\mathbf{R}^2$ (in the direction coherent to a complex orientation) or the Morse modification at infinity of the double point which associates to the double point of $\mathbf{R}^2$ two points of the real line $\mathbf{R}P^2\backslash\mathbf{R}^2$.

The next Lemma enlarges the preceding statement to generic curves of degree m, type I and genus g, and more generally to singular curves of degree m and type I with non-degenerate singular points.

LEMMA 2.3. *Let $\mathcal{A}_m$ be a singular curve of degree m and type I with non-degenerate singular points, then its real point set $\mathbf{R}\mathcal{A}_m \subset \mathbf{R}P^2$ is a partially regular curve deduced from a generic immersion of the circle S^1 into the plane $\mathbf{R}^2$ with $\frac{(m-1)(m-2)}{2} \leq n \leq \frac{(m-1)(m-2)}{2}+[\frac{m}{2}]$, double points and Whitney index $\frac{(m-1)(m-2)}{2}+1$ if and only if its set of singular points consists of at most $\frac{(m-1)(m-2)}{2}$ crossings.*

proof: It follows from an argument similar to the one of the proof of Lemma 2.1 of Chapter 5. Q.E.D

Lifting of a generic path in the space of generic immersion of the circle into the plane to $\mathbf{R}\mathcal{C}_m$. According to the Lemma 2.1 of Chapter 5, one can consider $\mathbf{R}\mathcal{H}_m$ and $\mathbf{R}\mathcal{A}_m$ as smoothed immersions of the circle into the plane with the same Whitney index. Let us denote $\mathbf{R}\tilde{\mathcal{H}}_m$, $\mathbf{R}\tilde{\mathcal{A}}_m$ the corresponding immersions. From the Whitney's Theorem, there exists a path $\tilde{h}$ which connects $\mathbf{R}\tilde{\mathcal{H}}_m$ and $\mathbf{R}\tilde{\mathcal{A}}_m$. In this section, we shall in Proposition 2.4 of Chapter 5 and Theorem 2.7 of Chapter 5 define a path S in $\mathbf{R}\mathcal{C}_m$ $S(0) = \mathbf{R}\mathcal{H}_m$,
$S(1) = \mathbf{R}\mathcal{A}_m$ from lifting a path $\tilde{h}$ which connects $\mathbf{R}\tilde{\mathcal{H}}_m$ and $\mathbf{R}\tilde{\mathcal{A}}_m$ in the space of immersions of the circle into $\mathbf{R}^2$.

By means of the Lemmas 2.1 of Chapter 5 and 2.3 of Chapter 5, we shall develop properties provided by the genus property and interpret the path $\tilde{h}$: $\mathbf{R}\tilde{\mathcal{H}}_m \to \mathbf{R}\tilde{\mathcal{A}}_m$ as a change of relative position of the 1-handles of $(\mathbf{C}\mathcal{H}_m) \approx S^2_{g_m}$, $(\mathbf{C}\mathcal{H}_m) \supset \mathbf{R}\mathcal{H}_m$ which gives $(\mathbf{C}\mathcal{A}_m) \approx S^2_{g_m}$, $(\mathbf{C}\mathcal{A}_m) \supset \mathbf{R}\mathcal{A}_m$. The path S appears as the track on $\mathbf{R}P^2$ of a diffeotopy h_t of $\mathbf{C}P^2$, $t \in [0,1]$, $h(0) = \mathbf{C}\mathcal{H}_m$, $h(1) = \mathbf{C}\mathcal{A}_m$.

(5 of Chapter 5).A Lifting Perestroikas in the space $\mathbf{R}\mathcal{C}_m$

Recall that given a perestroika along a path in the space $\mathcal{C}_{I,g}$ we call *smoothed perestroika* the change obtained by smoothing the fragments involved in the perestroika. We shall call *diffeotopic perestroika* of perestroika defined along a path h in the space $\mathcal{C}_{I,g}$) of irreducible curves of type I and genus g and degree $1 \leq n \leq m$, the change experienced by a curve in $\mathcal{C}_{I,g}$ with the following property.
For any point a which participated in a perestroika (along the path h) let $U(a)$ be a small ϵ-neighborhood around a of the complex point set of the initial curve embedded in $\mathbf{C}P^2$, (one can choose also $U(a)$ as follows: $U(a) = \{z =< u, p >= (u_0.p_0 : u_1.p_1 : u_2.p_2) \in \mathbf{C}P^2 \mid u = (u_0, u_1, u_2) \in U^3_{\mathbf{C}}, p = (p_0 : p_1 : p_2) \in D(a, \epsilon)\}$ where $D(a, \epsilon)$ denotes a small disc (in the Fubini-Study metric) of radius ϵ around a in $\mathbf{R}P^2$. Restrictions to $U(a)$ of the complex point set of the initial curve and of the complex point set which results from the diffeotopic perestroika are diffeotopic. (As will become clear later, in most cases, given a curve $\mathcal{A}_m$ the change experienced by the curve $\mathcal{A}_m$ after just one diffeotopic perestroika does not lead to a curve.)

Given a diffeotopic perestroika, we call *smoothed diffeotopic perestroika* the change obtained by smoothing the fragments involved in the diffeotopic perestroika (i.e fragments inside $U(a)$ for any a which participated in the perestroika).

(5 of Chapter 5).B Chain of Diffeotopic Perestroika

Let us recall that up to regular deformation of its polynomial (see Theorem 2.30 of Chapter 2), curve $\mathcal{H}_m$ results from the recursive construction of Harnack curves $\mathcal{H}_i$, $1 \leq i \leq m$, where $\mathcal{H}_{i+1}$ is deduced from classical small perturbation of the union $\mathcal{H}_i \cup L$ of the curve $\mathcal{H}_i$ with a line L.

Hence, when searching to interpret the path $\tilde{h} : \mathbf{R}\tilde{\mathcal{H}}_m \to \mathbf{R}\tilde{\mathcal{A}}_m$ as a change of relative position of the 1-handles of $(\mathbf{C}\mathcal{H}_m) \approx S^2_{g_m}$, $(\mathbf{C}\mathcal{H}_m) \supset \mathbf{R}\mathcal{H}_m$, one can proceed by induction and search for $1 \leq j \leq (m-1)$ to move the union of the g_j 1-handles of $\mathbf{C}\mathcal{H}_j \approx S^2_{g_j}$ with the $(j-1)$ more 1-handles of $\mathbf{C}\mathcal{H}_{j+1} \approx S^2_{g_j+(j-1)}$ in such a way that at the end of the process it results $(\mathbf{C}\mathcal{A}_m) \approx S^2_{g_m}$, $(\mathbf{C}\mathcal{A}_m) \supset \mathbf{R}\mathcal{A}_m$.

Using the recursive Morse-Petroskii's theory of chapter 4 as a tool to describe the lifting of $\tilde{h}$ as a track of a diffeotopy h_t of $\mathbf{C}P^2$, $t \in [0,1]$, $h(0) = \mathbf{C}\mathcal{H}_m$, $h(1) = \mathbf{C}\mathcal{A}_m$, we shall need *a global order* on the set of crossings of curves $\mathcal{H}_i \cup L$ and the associated k-uple.

a) global order

According to Theorem 3.9 of Chapter 2, for any Harnack curve $\mathcal{H}_m$ of degree m, there exists a finite number I ($I = \frac{m(m+1)}{2} + \Sigma_{k=2}^{k=[m/2]}(2k-3)$) of disjoint 4-balls $B(a_i)$ of radius ϵ (in the Fubini-Study metric) invariant by complex conjugation centered in points a_i of $\mathbf{R}P^2$ such that up to conj-equivariant isotopy of $\mathbf{C}P^2$, $\mathcal{H}_m \setminus \cup_{i \in I} B(a_i) = \cup_{i=1}^m L_i \setminus \cup_{i=1}^I B(a_i)$ where $L_1, ..., L_m$ are m distinct projective lines with

$$L_i \setminus \cup_{i=1}^I B(a_i) \cap L_j \setminus \cup_{i=1}^I B(a_i) = \emptyset$$

for any $i \neq j$, $1 \leq i, j \leq m$.

The proof of Theorem 3.9 of Chapter 2 is based on an induction on the degree m of the curve $\mathcal{H}_m$ which provides a natural order on the lines L_j, $1 \leq j \leq m$. Since any 4-ball $B(a_i)$) intersects exactly 2 lines L_{j-1}, L_j the natural order on the set of lines L_j extends to an order on the set of points a_i. We shall say that a_i has order j, $2 \leq j \leq m$, if and only if $B(a_i)$ intersects the lines L_{j-1} and L_j. (In other words, in the inductive construction, any crossing of $\mathcal{H}_j \cup L$ has order j; and according to Definition 2.1 of Chapter 4, the first point of a k-uple has the order of the crossing to which it is associated.)

Such order extends on 3-uple (resp the 2-uple) as follows. Given a_j of global order j the first point of a 3-point (a_j, a_{j+1}, a_{j+2}), (j even) associated to a a_{j+1} has global order $j+1$, a_{j+2} has global order $j+2$.

Consider the set a_j $1 \leq j \leq J = \frac{m(m+1)}{2}$ of crossings of curves $\mathcal{H}_i \cup L$, $1 \leq i \leq (m-1)$ in the inductive construction of $\mathcal{H}_m$. It may be easily easily extracted from the proof of Theorem 3.9 of Chapter 2 (see also Proposition 3.7 of Chapter 2 and Definition 2.1 of Chapter 4) that $\cup_{i=1}^I B(a_i) \supset \cup_{j=1}^{\frac{m(m+1)}{2}} U(a_j)$ where $B(a_i)$ is the 4-ball around a_i of radius ϵ and $U(a_j)$ is the ϵ-neighborhood around a_j. (According to terminology of section 2 of Chapter 2 (see proposition 3.7 of Chapter 2) the set of points $\cup_{i \in I} a_i = \cup_{n=1}^m A_n$ where A_n denotes the set of points perturbed in a maximal simple of deformation of $\mathcal{H}_n$.)

For sake of simplicity, according to corollary 3.5 of Chapter 2, we shall consider the T-inductive construction of Harnack curves and e-tubular neighborhoods $U(p)$ (see definition 2.2 of Chapter 4) defined from faces $\Gamma \subset l_j$, $l_j = \{(x,y) \in (\mathbf{R}^+)^2 | x+y=j\}$. As already introduced in subsection 5 of Chapter 5, $\mathcal{P}_m$ denotes the set of points p with the property: $\cup_{p\in\mathcal{P}_m} U(p) = \cup_{a_i\in I} U(a_i)$. (In such a way, we have the following equivalent definition of the global order for the patchworking construction. Any point a of the set $a_i \in I$ has order $j+1$, $2 \leq j+1 \leq m$, if and only if $a \in U(p)$ where $U(p)$ is the ϵ-tubular neighborhood defined from the interior Γ^0 of a face $\Gamma \subset l_j$, $l_j = \{(x,y) \in (\mathbf{R}^+)^2 | x+y=j\}$. This global order extends on k-uple as previously explained.)
Let us extend the global order on the set points $a_i, i \in I$ to the set $\cup_{a_i\in I} U(a_i)$. We shall say that $U(a_i)$ has order j if a_i has global order j. Equivalently, in the pathchworking scheme, $U(p) = U(a_i)$ has order j if it is defined from the interior Γ^0 of a face $\Gamma \subset l_{j-1}$.

b) Covering of $\mathbf{C}P^2$

The union $\cup_{i\in I} U(a_i)$ does not cover $\mathbf{C}P^2$. We shall consider consider the family $\tilde{U}(a_j)$, $\cup_{j=1}^J \tilde{U}(a_j) \supset \cup_{i=1}^I B(a_i)$, of neighborhoods of points a_j, $j \in J$, verifying the following properties:
- any $\tilde{U}(a_l)$ of order j intersects two neighborhoods $\tilde{U}(a_k)$ of order $j-1$,

$$\begin{aligned}
& \cup_{j=2}^m \cup_{a_i\ of\ order\ j} \tilde{U}(a_i) && = \mathbf{C}P^2 \\
-for\ j=2: & && \\
& \cup_{a_i\ of\ order\ 2} \tilde{U}(a_i) && \supset \mathbf{C}L_2 \setminus \cup_{a_i of order 3} \tilde{U}(a_i) \\
& \cup_{a_i\ of\ order\ 2} \tilde{U}(a_i) && \supset \mathbf{C}L_1 \\
-for\ 3 \leq j < m: & && \\
& \cup_{a_i\ of\ order\ j} \tilde{U}(a_i) && \supset \mathbf{C}L_j \setminus \cup_{a_i\ of\ order\ j+1} \tilde{U}(a_i) \\
& \cup_{a_i\ of\ order\ j} \tilde{U}(a_i) && \supset \mathbf{C}L_{j-1} \setminus \cup_{a_i\ of\ order\ j-1} \tilde{U}(a_i) \\
-for\ j=m: & && \\
& \cup_{a_i\ of\ order\ m} \tilde{U}(a_i) && \supset \mathbf{C}L_m \\
& \cup_{a_i\ of\ order\ m} \tilde{U}(a_i) && \supset \mathbf{C}L_{m-1} \setminus \cup_{a_m\ of\ order\ m-1} \tilde{U}(a_i)
\end{aligned}$$

Given $\mathcal{H}_m$ the Harnack curve of degree m and $\mathcal{A}_m$ a smooth curve of type I, we shall in the Proposition 2.4 of Chapter 5, define a path

$$S : [0,1] \to \mathbf{R}\mathcal{C}_m$$

$S(0) = \mathbf{R}\mathcal{H}_m$, $S(1) = \mathbf{R}\mathcal{A}_m$ described locally in the opens $\tilde{U}(a_i)$ and sequentially by induction on the global order on the set points a_i,$i \in I$ and the associated k-uple by diffeotopic perestroikas of perestroikas in the spaces $\mathcal{C}_{I,g}$ of irreducible curves of degree $1 \leq n \leq m$, type I and genus g. Besides, any singular point which participated in diffeotopic perestroika belongs to $\cup_{i=1}^I B(a_i)$.

PROPOSITION 2.4. *Let $\mathcal{H}_m$ be the Harnack curve of degree m and $\mathcal{A}_m$ be a smooth curve of type I. There exists a path*

$$S : [0,1] \to \mathbf{R}\mathcal{C}_m, S(0) = \mathbf{R}\mathcal{H}_m, S(1) = \mathbf{R}\mathcal{A}_m$$

which crosses the discriminant hypersurface $\mathbf{R}\mathcal{D}_m$. Up to conj-equivariant isotopy of $\mathbf{C}P^2$, the path may be sequentially defined in the open $\tilde{U}(a_i)$ by induction on the global order by smoothed diffeotopic perestroikas of diffeotopic perestroikas on the spaces $\mathcal{C}_{I,g}$ of curves of type I, degree $1 \leq n \leq m$, and genus g as follows:

The sequence of diffeotopic perestroika is defined as follows:
Assume $\mathcal{H}_m$ obtained via the patchworking method.

(1) *one can assume that only points $\cup_{i \in I} a_i$ are double-points involved in a diffeotopic perestroika.*
(2) *any diffeotopic perestroika of which real ordinary double-point are points of order at most j is a diffeotopic perestroika in the space $\mathcal{C}_{I,g}$ of curves of degree j, type I and genus g.*
(3) *a diffeotopic perestroika defines double-points involved in next as follows. As branches involved in the topological meaning of the critical point associated to a real ordinary double-point of order j involved in the diffeotopic perestroika move under the diffeotopic perestroika they define real ordinary double-point of order $j+1$ involved in a next diffeotopic perestroika. in the space $\mathcal{C}_{I,g}$ of curves of degree $j+1$, type I and genus g.*
Any imaginary point which participated to such sequence of diffeotopic perestroika belongs to the intersection of a neighborhood $\tilde{U}(a_j)$ of a point order j and a neighborhood of a point $\tilde{U}(a_{j+1})$ of order $j+1$. Besides, imaginary points participated in such a way that if one α-point (or β-point) appears (resp, disappears) after a diffeotopic perestroika on real points of order j, then it disappears (resp, appears) after a diffeotopic perestroika on real points of order $j+1$.

proof:

Let us explain the method of our proof.

We shall prove that up to slightly modify the coefficients of the polynomial giving the curve $\mathcal{A}_m$, one can always assume that there exists a diffeotopy h_t of $\mathbf{C}P^2$ $h(0) = \mathbf{C}\mathcal{H}_m$, $h(1) = \mathbf{C}\mathcal{A}_m$ with the property $h_t(\mathbf{C}\mathcal{A}_m \cap U(p)) \subset U(p)$ for any $p \in \mathcal{P}_m$.

Our argumentation is based on the Lemma 2.1 of Chapter 5 and its proof. According to the Lemma 2.1 of Chapter 5, one can consider $\mathbf{R}\mathcal{H}_m$ and $\mathbf{R}\mathcal{A}_m$ as smoothed immersions of the circle into the plane with the same Whitney index. We shall denote $\mathbf{R}\tilde{\mathcal{H}}_m$, $\mathbf{R}\tilde{\mathcal{A}}_m$ the corresponding immersions. From the Whitney's theorem, there exists a path $\tilde{h}$ which connects $\mathbf{R}\tilde{\mathcal{H}}_m$ and $\mathbf{R}\tilde{\mathcal{A}}_m$.

Therefore, the definition of the path S in $\mathbf{R}\mathcal{C}_m$ $S(0) = \mathbf{R}\mathcal{H}_m$, $S(1) = \mathbf{R}\mathcal{A}_m$ may be deduced from a lifting of the path $\tilde{h}$ which connects $\mathbf{R}\tilde{\mathcal{H}}_m$ and $\mathbf{R}\tilde{\mathcal{A}}_m$ in the space of immersions of the circle into $\mathbf{R}^2$.

Since any smooth curve is irreducible, and any reducible polynomial is the product of a finite number of reducible polynomials, we shall, according to Lemma 2.1 of Chapter 5 and Lemma 2.3 of Chapter 5, lift the path $\tilde{h} : \mathbf{R}\tilde{\mathcal{H}}_m \to \mathbf{R}\tilde{\mathcal{A}}_m$ in the space of immersion of the circle into the plane from smoothed diffeotopic perestroikas of diffeotopic perestroikas in the spaces $\mathcal{C}_{I;g}$ of curves of degree $n \leq m$, type I and genus $0 \leq g \leq \frac{(n-1).(n-2)}{2}$.

For any diffeotopic perestroika π experienced along a generic path in $\mathcal{C}_{I;g}$, locally a description of the real components of the resulting curve is deduced from double points involved in the diffeotopic perestroika. Thus, any smoothed diffeotopic perestroika π experienced along S, may be defined from its smoothed double points.
From the detailed construction of $\mathcal{H}_m$ given in chapter 4, we shall get smoothed points of smoothed diffeotopic perestroika experienced along the path S.
As already introduced in the proof of Lemma 2.1 of Chapter 5, we shall consider h the diffeomorphism $h : \mathbf{C}\mathcal{H}_m \to S^2_{g_m} \subset \mathbf{R}^3$ and $r : \mathbf{R}^3 \to \mathbf{R}^2$ the projection which maps $h(\mathbf{R}\mathcal{H}_m)$ to $r(h(\mathbf{R}\mathcal{H}_m) = \mathbf{R}\tilde{\mathcal{H}}_m \subset \mathbf{R}^2$. We shall prove that only points of $\mathbf{R}\tilde{\mathcal{H}}_m$ smoothed in $\mathbf{R}^2$ to give $\mathbf{R}\mathcal{H}_m$ may lift to smoothed double points of an irreducible curve of degree n, $1 \leq n \leq m$ type I and genus g involved in smoothed diffeotopic perestroika. Thus, double-points of $\mathbf{R}\tilde{\mathcal{H}}_m$ smoothed at infinity are involved along the path $\tilde{h} : \mathbf{R}\tilde{\mathcal{H}}_m \to \mathbf{R}\tilde{\mathcal{A}}_m$ as the relative position of the 1-handles of $(\mathbf{C}\mathcal{H}_m) \approx S^2_{g_m}$, $(\mathbf{C}\mathcal{H}_m) \supset \mathbf{R}\mathcal{H}_m$ changes to give $(\mathbf{C}\mathcal{A}_m) \approx S^2_{g_m}$, $(\mathbf{C}\mathcal{A}_m) \supset \mathbf{R}\mathcal{A}_m$.

In such a way, using the path provided by Lemma 2.1 of Chapter 5 and Whitney's theorem, according to the Lemma 2.3 of Chapter 5 and the fact that $\mathbf{C}\mathcal{H}_m$ and $\mathbf{C}\mathcal{A}_m$ are diffeotopic, we shall characterize the track S on $\mathbf{R}P^2$ of a diffeotopy h_t of $\mathbf{C}P^2$, $t \in [0,1]$, $h(0) = \mathbf{C}\mathcal{H}_m$, $h(1) = \mathbf{C}\mathcal{A}_m$.

We shall divide our proof into three parts:

1. In the first part, we define double-points of curves along S.
2. The second part deals with smoothed diffeotopic perestroikas along S.
3. In the third part, we give a method which provides any curve $\mathcal{A}_m$ of degree m and type I from $\mathcal{H}_m$, that is $\mathbf{R}\mathcal{A}_m$ from $\mathbf{R}\mathcal{H}_m$.

The first two parts make use of the idea that locally a real branch does not differ from another real branch. In these parts, we consider the situation more locally than globally and fix definitions we shall need in the third part. In the third part, we give a method of organizing these local moves in such a way that globally it results a real algebraic curve in $\mathbf{C}P^2$.

The exposition of the main ideas of our proof is now finished, and we shall proceed to precise arguments.

1. Double points of curves along S.

Let us start our definition of double points of curves along the path S we have planed to construct by the following remark. Assume $\mathcal{H}_m$ constructed via the patchworking method. Given $\mathcal{H}_m$, and its projection $r(h(\mathbf{R}\mathcal{H}_m)) = \mathbf{R}\tilde{\mathcal{H}}_m$, it is easy to deduce from the Lemma 2.1 of Chapter 5 and its proof and Lemma 2.3 of Chapter 5, that for $m > 2$ only the g_m points of $\mathbf{R}\tilde{\mathcal{H}}_m$ smoothed in $\mathbf{R}^2$ to give $\mathbf{R}\mathcal{H}_m$ may lift to real double points of curves along S.

Denote
$\rho^m : (T_m \times U^2_{\mathbf{C}}) \to \mathbf{C}T_m \approx \mathbf{C}P^2$ the natural surjection. According to the chapter 2 (see Proposition 3.7 of Chapter 2 and Theorem 3.9 of Chapter 2), the Harnack curve $\mathcal{H}_m$ of degree m is obtained from recursive perturbation of curves $\mathcal{H}_{m-i} \cup L$, $0 < i < m$, (with $L \approx \mathbf{C}l_{m-i} = \rho^m(l_{m-i} \times U^2_{\mathbf{C}})$) in such way that outside a finite number I of 4-ball $B(a_i) \subset \mathbf{C}P^2$ there exists a conj-equivariant isotopy j_t of subset of $\mathbf{C}P^2$ which maps:

(1) $\mathcal{H}_{m-i}\backslash \cup_{i=1}^{I} B(a_i)$ onto a part of the curve $\mathcal{H}_{m-i+1}$.
(in the patchworking scheme, such part is contained in the restriction $\rho^m(T_{m-i} \times U^2_{\mathbf{C}})$ of $\mathbf{C}P^2 \approx \mathbf{C}T_m$.)
(2) $(\mathcal{H}_{m-i}\backslash j_1(\mathcal{H}_{m-i+1}\backslash \cup_{i=1}^{I} B(a_i))$ onto a part of the projective line L.
(in the patchworking scheme, such part is contained in the restriction $\rho^m(D_{m-i+1,m-i} \times U^2_{\mathbf{C}})$ of $\mathbf{C}P^2 \approx \mathbf{C}T_m$.)

Consequently, (see Theorem 3.9 of Chapter 2), outside a finite number of 4-balls $B(a_i)$, the curve $\mathcal{H}_m$, is up to conj-equivariant isotopy of $\mathbf{C}P^2$, the union of m projective lines minus their intersections with 4-balls $B(a_i)$.

Since projective lines are diffeotopic to circles, outside a finite number of 4-balls $B(a_i)$, components of $\mathbf{R}\mathcal{H}_m$ can be considered as components of m concentric circles. Thus, smoothed real points involved in diffeotopic perestroikas along S belong to the union of real components of $\mathcal{H}_m$ contained in $\rho^m(D_{m-i+1,m-i} \cup D_{m-i,m-i-1} \times U^2_{\mathbf{C}})$ where $D_{m-i+1,m-i}$ and $D_{m-i,m-i-1}$ are bands of T_m (for $j \in \{m-i-1, m-i\}$, $D_{j+1,j} = \{x \geq 0, y \geq 0 | j \leq x+y \leq j+1\}$) with intersection $D_{m-i+1,m-i} \cap D_{m-i,m-i-1} = l_{m-i}$.

Therefore, locally any smoothed real point can be considered as the smoothed intersection of two real branches of $\mathbf{R}\mathcal{H}_m \cap \rho^m(D_{m-i+1,m-i} \cup D_{m-i,m-i-1} \times U^2_{\mathbf{R}})$ of which intersection point belongs to a neighborhood of $\rho^m(l_{m-i} \times U^2_{\mathbf{R}})$.

Consequently, it is easy to deduce from relative orientation and relative location of real branches of $\mathbf{R}\mathcal{H}_m \cap \rho^m(D_{m-i+1,m-i} \cup D_{m-i,m-i-1} \times U^2_{\mathbf{C}})$ that smoothed real points involved in diffeotopic perestroika along S belong to $\rho^m(D(p) \times U^2_{\mathbf{R}}) = U(p) \cap \mathbf{R}^2$ with $p \in \mathcal{P}_m$ and that any $U(p)$ contains one such smoothed double-point.

In such a way, according to Lemma 2.1 of Chapter 5 and Lemma 2.3 of Chapter 5, smoothed real points along S are smoothed crossings which belong to the set
$\cup_{p \in \mathcal{P}_m} \rho^m(D(p) \times U^2_{\mathbf{R}}) = \cup_{p \in \mathcal{P}_m} U(p) \cap \mathbf{R}^2$.

2.Smoothed diffeotopic perestroika on smoothed double points of curves along S.

Recall that from the previous Morse-Petrovskii's study of Harnack curves (see chapter 4) for any $p \in \mathcal{P}_m$, there exists a crossing $a \in \mathcal{H}_{m-i} \cup L$ with $a \in U(p) = \rho^m(D(p) \times U^2_{\mathbf{C}})$ such that the crossing $a \in \mathcal{P}_m$ is associated to:

(1) either a simple-point of $\mathcal{H}_m$
(2) either a 3-point $(p_{m-i+1}, p_{m-i+2}, p_{m-i+3})$ (where $p_{m-i+3} \approx p_m$ of $\mathcal{H}_m$
(3) or in case m odd a 2-point (p_{m-1}, p_m) of $\mathcal{H}_m$

As recalled previously, cusp perestroika and weak triple point perestroika do not change real and complex smoothed parts. Therefore, there is not need to consider them. Besides, since smoothed real points of generic curves along S are smoothed crossings, no smoothed imaginary self-tangency perestroika is undergone along S. (This restriction is not in contradiction with the fact that $\mathcal{A}_m$ is an arbitrary curve of degree m and type I since $\mathcal{H}_m$ is an M-curve.)

Obviously, in $U(p)$ any change implied by a smoothed diffeotopic perestroika is a perturbation in which, besides real branches of the crossing $a \in U(p)$ of $\mathcal{H}_j \cup L$,

real branches involved in the local topological meaning of simple point and of any point of a 3-point (or 2-point in case m odd) participated.
We shall study the situations inside the four 4-balls with non-empty real part and globally invariant by complex conjugation of $U(p) \subset (\mathbf{C}^*)^2$ simultaneously.

We shall distinguish perestroikas in which only simple points are involved and perestroikas in which k-points are involved.

2.a Smoothed diffeotopic perestroikas defined on simple points.

Let us consider smoothed diffeotopic perestroikas defined on simple points.

Recall that given a a crossing of $\mathcal{H}_j \cup L$, we say that p_l the critical point of $\mathcal{H}_l$ $j+1 \leq l \leq m$, associated to a is simple point if:

(1) $l \geq j+1$,
(2) any critical point p_k, $j+1 \leq k \leq l$ is equivalent to p_{j+1}.

Given a a crossing of $\mathcal{H}_j \cup L$ ($L \approx \mathbf{C}l_j$) $1 \leq j \leq m$ denote (x_0, y_0) the simple point of $\mathcal{H}_m$, associated the crossing a. Let $p \in \mathcal{P}_m$, such that $a, (x_0, y_0) \in \rho^m(D(p) \times U^2_{\mathbf{R}}) = U(p) \cap (\mathbf{R}^*)^2$ Real branches in $U(p)$ are the real branches involved in the topological meaning of (x_0, y_0) and the real branches which result from the perturbation of the crossing a. Let us remark that if the relative position of real branches which result from the perturbation of the crossing a is preserved, then relative position of the real real branches involved in the topological meaning of (x_0, y_0) is also preserved. Otherwise, according to Lemmas 2.6 of Chapter 4, 2.7 of Chapter 4, 2.9 of Chapter 4 and Lemma 2.1 of Chapter 5, it would lead with contradiction with the fact that we consider only curves of type I that is only Morse modification in the direction coherent with complex orientation.

As already noticed, in most perestroikas imaginary points are involved. Considering together imaginary and real points involved in these perestroikas, we shall define smoothed diffeotopic perestroikas of perestroika along a path h in the space $\mathcal{C}_{I,g}$) (see section 5 of Chapter 5) and get relative position of real branches in $U(p)$ after such diffeotopic perestroikas.

Lemma 2.5. *Let π be either a direct or an inverse self tangency smoothed diffeotopic perestroika or a triple point with imaginary branches smoothed diffeotopic perestroika along the path*

$$S : [0,1] \to \mathbf{R}\mathcal{C}_m$$

$S(0) = \mathbf{R}\mathcal{H}_m$, $S(1) = \mathbf{R}\mathcal{A}_m$. Let $c \in \rho^m(D(p) \times U^2_{\mathbf{R}}) = U(p) \cap (\mathbf{R}^2)$, with $p \in \mathcal{P}_m$ (which may be a or (x_0, y_0)) a real point involved in π.

Then, under π relative position of real branches in $U(p)$ changes as follows. One can always assume that the crossing $a \in \mathcal{H}_j \cup L$ is the real (smoothed) point involved in the perestroika, the relative position of real branches involved in the local topological meaning of (x_0, y_0) is changed as follows. Draw a dotted (unknotted) line between the two real branches involved in the local topological meaning of the critical point (x_0, y_0) The diffeotopic perestroika moves the real component around a according to the perestroika and leaves the two real branches linked by the unknotted dotted line.

proof:

(a)

Let π be a direct self tangency perestroika or inverse self tangency perestroika in the negative direction.

Denote (x_0, y_0) the critical point of B_{j+1} associated to the crossing a such that $a, (x_0, y_0) \in \rho^m(D(p) \times U^2_{\mathbf{R}}) = U(p) \cap (\mathbf{R}^*)^2$.

Denote c, c' the two real points involved in π.
Let $p, p' \in \mathcal{P}_m$ such that $c \in \rho^m(D(p) \times U^2_{\mathbf{R}}) = U(p) \cap \mathbf{R}^2$,
$c' \in \rho^m(D(p') \times U^2_{\mathbf{R}}) = U(p') \cap \mathbf{R}^2$ Then, imaginary points i, i' involved in π are such that: $i \in \rho^m(D(p) \times U^2_{\mathbf{C}}) = U(p)$,
$i' \in \rho^m(D(p') \times U^2_{\mathbf{C}}) = U(p')$.

In an ϵ-neighborhood $U(p)$, $p \in \mathcal{P}_m$, the complex point set of the Harnack curve $\mathcal{H}_{j+1}$ can be considered as the image of a smooth section of a tubular fibration $N \to (\mathbf{C}V \setminus \{a\}))$ where $V = (\mathcal{H}_{j+1} \cup L) \cap U(p)$ (i.e, from Morse Lemma, in a neighborhood of a, V looks like the intersection of two real lines in the point a) and N is an ϵ-tubular neighborhood of V.

Assume $c = a$, (resp $c = (x_0, y_0)$), relative location of the real branches in $U(p)$ is changed after diffeotopic perestroika in such a way that after diffeotopic perestroika the complex point set is the image of a smooth section of a tubular fibration $N \to (\mathbf{C}V \setminus \{a\}))$. Therefore, since (x_0, y_0) (resp, a) is not a real point involved in the perestroika, the complex point i lies in complex 4-ball around (x_0, y_0) which contains real branches involved in the topological meaning of (x_0, y_0) (resp, which results from the perturbation of the singular crossing a). Hence, it follows the relative position of the real branches in $U(p)$. Obviously, the relative position of the real branches inside $U(p')$ may be deduced from the same argument.

(b)
Consider now π a triple point perestroika with imaginary branches. Obviously, triple point with imaginary branches perestroika in the positive (resp, negative) direction may appear only after direct (resp, inverse) self tangency perestroika. Such perestroika provides two imaginary smoothed points, each of which belongs to an ϵ-neighborhood $U(p)$ $p \in \mathcal{P}_m$.

Assume an oval $\mathcal{O}$ is involved in a smoothed triple point perestroika with imaginary branches. Then, $\mathcal{O}$ intersects $\cup_{p \in \mathcal{P}_m} U(p)$ in four of its neighborhood $U(p)$. Each of these four neighborhoods $U(p)$ contains one imaginary point involved in the perestroika: two of them contain one of the two imaginary points required before the perestroika, and the two others contain one of the two imaginary points which appear after the perestroika.

Q.E.D

2.b Smoothed diffeotopic perestroikas defined on 2-points and 3-points.

We shall now consider crossings associated to 3-point of $\mathcal{H}_m$ and 2-point in case of odd m in the construction of $\mathcal{H}_m$. Let us prove the following Lemma.

LEMMA 2.6. *A smoothed strong real triple point diffeotopic perestroika π may be experienced along the path*

$$S : [0, 1] \to \mathbf{R}\mathcal{C}_m$$

$S(0) = \mathbf{R}\mathcal{H}_m$, $S(1) = \mathbf{R}\mathcal{A}_m$ *only in a neighborhood which contains a crossing $a \in \mathcal{H}_j \cup L$ associated to a triple-point or a double-point of the curve $\mathcal{H}_m$.*

proof:

Recall that given a a crossing of one curve $\mathcal{H}_{j-1} \cup L$ (with j even) associated to a 3-uple (p_j, p_{j+1}, p_{j+2}) of $\mathcal{H}_{j+2}$, $j+2 \leq m$, we consider the following natural order on the 3-uple (p_j, p_{j+1}, p_{j+2}): p_j has order j, p_{j+1} has order $j+1$, p_{j+2} has order $j+2$.

1Let us first consider the 2-uple (p_j, p_{j+1}).
1.a Let $b_j(p_j) = c_0$ with (p_j, p_{j+1}) a 2-uple of the first or the second kind. It follows from Lemma 2.7 of Chapter 4 of chapter 4 that in case of 2-uple of the first kind: when $c = c_0$, the non-empty positive oval touches a positive outer oval; and from Lemma 2.9 of Chapter 4 of chapter 4 that in case of 2-uple of the second kind: when $c = c_0$, the non-empty positive oval touches a negative outer oval.
1.b Let $b_{j+1}(p_{j+1}) = c_0$ with (p_j, p_{j+1}) a 2-uple of the first or the second kind. When $c = c_0$, the one-side component of the curve $\mathcal{H}_{j+1}$ touches itself. Then, as c increases from c_0 to $c_0 + \epsilon$, one negative oval $\mathcal{O}^-$ disappears.

Besides, when consider together local topological meaning of p_j and p_{j+1} three real branches are involved which belong respectively to a part of $\mathcal{H}_{j-1}$ isotopic to the real line $L_{j-1} \approx \mathbf{R}l_{j-2}$, a part of $\mathcal{H}_j$ isotopic to the real line $L_j \approx \mathbf{R}l_{j-1}$, a part of $\mathcal{H}_{j+1}$ isotopic to the real line $L_{j+1} \approx \mathbf{R}l_j$. Besides, the branches which are respectively isotopic to a part of the line $L_j \approx \mathbf{R}l_{j-1}$ and isotopic to a part of the line $L_{j+1} \approx \mathbf{R}l_j$ belong to an oval $\mathcal{O}$ (negative in case of 2-uple of the first kind and positive in case of 2-uple of the second kind) which results from the perturbation of $\mathcal{H}_j \cup L$, $L_{j+1} \approx \mathbf{C}l_j$, around two crossings.

Consider $\mathbf{C}\mathcal{H}_{j+1}$ inside $\rho^m(D_{j,j-1} \cup D_{j+1,j} \times U^2_{\mathbf{R}}) \subset \mathbf{R}P^2$ where
$D_{j+1,j} = \{x \geq 0, y \geq 0 | j \leq x + y \leq j+1\}$,
$D_{j,j-1} = \{x \geq 0, y \geq 0 | j-1 \leq x + y \leq j+1\}$.

It is an easy consequence of the Lemma 2.7 of Chapter 4 and 2.9 of Chapter 4 of chapter 4 that one can consider the change from p_j to p_{j+1} as a "jump" of the branch of $\mathcal{H}_{j-1}$ isotopic to the real line $L \approx \mathbf{R}l_{j-2}$ from $\rho^m(D_{j,j-1} \times U^2_{\mathbf{R}})$ to $\rho^m(D_{j+1,j} \times U^2_{\mathbf{R}})$, namely as a smoothed triple point perestroika.

2 Consider now 3-uple (p_j, p_{j+1}, p_{j+2}) of the first and second kind.
Let $b_{j+2}(p_{j+2}) = c_0$,
2.a
in case of 3-uple of the first kind, it follows from Lemma 2.7 of Chapter 4 of chapter 4, that when $c = c_0$ one positive outer oval of the curve $\mathcal{H}_{j+2}$ touches another outer positive oval;
2.b
in case of 3-uple of the first kind, it follows from Lemma 2.9 of Chapter 4 of chapter 4, that when $c = c_0$ one negative inner oval of the curve $\mathcal{H}_{j+2}$ touches another negative oval.

It is obvious that the previous description of the strong triple point perestroika defined on the 2-uple (p_j, p_{j+1}) extends to the definition of a strong triple point perestroika defined on the 3-uple (p_j, p_{j+1}, p_{j+2}) in such a way that relative position and orientation of the two real branches involved in the topological meaning of p_{j+2} does not change under the perestroika.

Therefore, the Lemma is straightforward consequence of the previous study of the 2-uple and 3-uple. The varieties $\mathbf{C}\mathcal{H}_m$ and $\mathbf{C}\mathcal{A}_m$ are diffeotopic; under the perestroika the diffeotopy is preserved locally in any neighborhood $U(p)$, $p \in \mathcal{P}_m$.

Q.E.D

As explained in section 5 of Chapter 5, any smoothed perestroika π required a given location and orientation of real branches. In particular, smoothed triple-point perestroika is possible only in a neighborhood of an oval and of three real branches close to this oval with relative location and orientation as described in section 5 of Chapter 5. Namely, the real branches involved in the local topological meaning of p_j may be involved in a triple point diffeotopic perestroika only if they have been reproached by previous perestroikas. It is easy to deduce from the relative location of branches involved in the topological meaning of critical point that in case of 2-uple of the first kind, the singular situation $b_{j+1}(p_{j+1}) = c_0$ is possible if and only if two positive outer ovals coalesce with the one-side component of the curve. In case of 2-uple of the second kind, the singular situation $b_{j+1}(p_{j+1}) = c_0$ is possible if and only if two negative inner ovals coalesce with the one-side component of the curve. We shall precise this remark in the third part.

3. Method which provides $(\mathbf{C}P^2, \mathbf{C}\mathcal{A}_m)$ from $(\mathbf{C}P^2, \mathbf{C}\mathcal{H}_m)$

Denote $\mathcal{S}_{I,m}$ the set $\mathcal{C}_{I,m} \backslash \mathcal{D}_m$ of smooth curves of degree m and type I.

We shall define a family of moves $\phi : \mathcal{H}_m \to \mathcal{S}_{I,m}$ with the property that in these moves the set $\cup_{i=1}^{I} a_i = \cup_{n=1}^{m} A_n$ of crossing-points perturbed in the construction of $\mathcal{H}_m$ participate as follows. Moves of real components of $\mathcal{H}_m$ correspond to an other choice of perturbation on these crossings.

Briefly the method of our proof is the following:

We shall apply successively Lemma 2.5 of Chapter 5 and 2.6 of Chapter 5 inside the sets:

$$\rho^m((D_{1,0} \cup D_{2,1}) \times U^2_{\mathbf{C}})$$

$$\rho^m((D_{2,,1} \cup D_{3,2}) \times U^2_{\mathbf{C}})$$
$$\rho^m((D_{3,2} \cup D_{2,1}) \times U^2_{\mathbf{C}})$$
$$\ldots$$
$$\rho^m((D_{m-i+1,m-i} \cup D_{m-i+2,m-i+1}) \times U^2_{\mathbf{C}})$$
$$\rho^m((D_{m-i+3,m-i+2} \cup D_{mm-i+2,m-i+2}) \times U^2_{\mathbf{C}})$$
$$\ldots$$
$$\rho^m((D_{m-1,m-2} \cup D_{m-2,m-3}) \times U^2_{\mathbf{C}})$$
$$\rho^m((D_{m,m-1} \cup D_{m-1,m-2}) \times U^2_{\mathbf{C}})$$

to describe the track S on $\mathbf{R}P^2$ of a diffeotopy h_t of $\mathbf{C}P^2$, $t \in [0,1]$, $h(0) = \mathbf{C}\mathcal{H}_m$, $h(1) = \mathbf{C}\mathcal{A}_m$ with $\mathcal{A}_m$ a curve of type I and degree m.

We shall now proceed to precise argument.

3.a Preliminaries

Recall that, according to chapter 4, for any point a perturbed in the construction of $\mathcal{H}_m$, there exists $p \in \mathcal{P}_m$ (where $\sharp\mathcal{P}_m = \frac{m(m-1)}{2}$) such that $a \in U(p)$ where

$U(p)$ is the ϵ-tubular neighborhood defined from the interior Γ^0 of a face Γ of the triangulation of T_m. According to Lemma 2.1 of Chapter 5 $\mathbf{R}\mathcal{H}_m$ (resp, $\mathbf{R}\mathcal{A}_m$) is a smoothed generic immersion of the circle S^1 into the plane $\mathbf{R}^2$, $\mathbf{R}\tilde{\mathcal{H}}_m$ (resp, $\mathbf{R}\tilde{\mathcal{A}}_m$) with $n \geq \frac{(m-1)(m-2)}{2}$ double points and Whitney index $\frac{(m-1)(m-2)}{2}+1$. The smoothing is such that $\frac{(m-1)(m-2)}{2}$ double points are smoothed in $\mathbf{R}^2$ by Morse modification in the direction coherent to the complex orientation. We shall denote by $\mathcal{G}_m$ the set of the $g_m = \frac{(m-1)(m-2)}{2}$ double points of $\mathbf{R}\tilde{\mathcal{H}}_m$ smoothed in $\mathbf{R}^2$ by Morse modification in the direction coherent to the complex orientation.

According to the part 1, we shall assume that the set $\mathcal{G}_m$ of double-points of $\mathbf{R}\tilde{\mathcal{H}}_m$ has the following properties:

(1) any two distinct points of $\mathcal{G}_m$ belong to two distinct neighborhoods $U(p)$ with $p \in \mathcal{P}_m$
(2) there exists a bijective correspondence between points of $\mathcal{G}_m$ and the g_m ovals $\mathcal{O}$ of $\mathcal{H}_m$ which do not intersect the line at infinity.

Properties (1) and (2) are straightforward consequences of the patchworking construction (see for example Corollary 2.4 of Chapter 1 and its proof) that any oval $\mathcal{O}$ of $\mathcal{H}_m$ which does not intersect the line at infinity, intersects two neighborhoods $U(p)$ defined from a face $\Gamma \subset l_j$,$1 \leq j \leq m-1$, $l_j = \{(x,y) \in (\mathbf{R}^+)^2 | x+y=j\}$, namely two neighborhoods $U(p)$ of global order $j+1$.

Besides, an oval $\mathcal{O}$ may intersect:

(1) either 4 neighborhoods $U(p)$ of the union $\cup_{p\in\mathcal{P}_m} U(p)$ such that two neighborhoods $U(p)$ are of global order j, $3 \leq j \leq m-1$, the two others are of global order $j-1$, $j+1$
(2) or 3 neighborhoods $U(p)$ of the union $\cup_{p\in\mathcal{P}_m} U(p)$ such that two neighborhoods $U(p)$ are of global order m, and the other is of global order $m-1$.

Therefore, one can choose the set $\mathcal{G}_m$ such that any point of $\mathcal{G}_m$ belongs to a neighborhood $U(p)$, $p \in \mathcal{P}_m$ of global order j, $2 \leq j \leq m-1$.

Consider the family $\tilde{U}(p) \supset U(p)$, $p \in \mathcal{P}_m$ $\cup_{p\in\mathcal{P}_m} \tilde{U}(p) = \mathbf{C}P^2$ verifying the properties (16), (16), (16), (16). Using the patchworking construction, according to the moment map $\mu : \mathbf{C}T_m \to T_m$ and the natural surjection $\rho^m : \mathbf{R}^+ T_m \times U^2_{\mathbf{C}} \to \mathbf{C}T_m$ (see the preliminary section), one can choose for example:

(1) for any $U(p)$ of global order $2 \leq j \leq m-1$, $\tilde{U}(p) = \rho^m(S \times U^2_{\mathbf{C}})$ where S is the unique square S with vertices $(c,d), (c+1,d), (c,d+1), (c+1,d+1)$, $c+d+1=j$ such that $U(p) \subset \tilde{U(p)}$.
(2) for any $U(p)$ of global order m, $\tilde{U}(p) = \rho^m(K \times U^2_{\mathbf{C}})$ where K is the polygon with vertices $(c,d), (c+2,d), (c,d+2)$, $c+d+1 = m-1$ such that $U(p) \subset \tilde{U}(p)$.

3.b Statement of the Method

We shall now deduce a lifting of the path $\tilde{h}$ in the space of immersions of the circle into $\mathbf{R}^2$ with index $g_m = \frac{(m-1)(m-2)}{2}$ in the space $\mathbf{R}\mathcal{C}_m$ and describe the topological pair $(\mathbf{C}P^2, \mathbf{C}\mathcal{A}_m)$. Such lifting is defined by smoothed diffeotopic perestroikas on $\mathbf{R}\mathcal{H}_m$ inside $\cup_{p\in\mathcal{P}_m} \tilde{U}(p)$.

Recall that, according to parts 1 and 2 of our proof, we shall consider only direct or inverse self tangency perestroika, triple point with imaginary branches perestroika, strong real triple point perestroika. Obviously, in any smoothed diffeotopic perestroika π undergone along the lifting S of $\tilde{h} : \mathbf{R}\tilde{\mathcal{H}}_m \to \mathbf{R}\tilde{\mathcal{A}}_m$, at least one oval $\mathcal{O}$ of $\mathbf{R}\mathcal{H}_m$ which does not intersect the line at infinity is involved. As already noticed, in most perestroikas imaginary points are involved. We shall consider the function J counting increment and decrement of α-point (double point of different halves) and β-point (double point of one of the halves) along the path $S : [0, 1] \to \mathbf{R}\mathcal{C}_m$.

Let $\mathcal{O}$ be an oval of $\mathcal{H}_m$. It intersects $\cup_{p\in\mathcal{P}_m}\tilde{U}(p)$ in four neighborhood $\tilde{U}(p)$, $p \in \mathcal{P}_m$: $U(p_1)$ of global order i with $2 \le i \le m-1$, , $U(p_2)$ and $U(p_3)$ of global order $i+1$, $U(p_4)$ of global order $i+2$.

We shall, from the Lemma 2.5 of Chapter 5 and 2.6 of Chapter 5, get moves of the real branches contained in $U(p_i)$, $1 \le i \le 4$, under any smoothed diffeotopic perestroika π.

We shall prove that smoothed diffeotopic perestroikas on the real components of $\mathcal{H}_m$ of which smoothed points are points of global order $2 \le j \le m-1$ (in the union of neighborhoods $\tilde{U}(p)$, $p \in \mathcal{P}_m$ of global order j, $2 \le j \le m-1$), induce smoothed diffeotopic perestroikas on the real component of $\mathcal{H}_m$ in neighborhoods $\tilde{U}(p)$ $p \in \mathcal{P}_m$ of global order $j+1$.
We shall call this process the step j of the lifting, and define the lifting of $\tilde{h}$: $\mathbf{R}\tilde{\mathcal{H}}_m \to \mathbf{R}\tilde{\mathcal{A}}_m$ by induction on the global order on the set of points $a_i, i \in I$ and the associated k-uple.

1) Let us first consider direct and inverse self-tangency and triple point with imaginary branches diffeotopic perestroika. According to Lemma 2.5 of Chapter 5 one can assume, without loss of generality that the smoothed real points involved in triple point with imaginary branches diffeotopic perestroika are the two crossings which belong to the neighborhoods $U(p_2)$, $U(p_3)$ of global order $i \le m-1$. In such a way, according to the Lemma 2.5 of Chapter 5, we get description of relative position of real branches in $U(p_2)$ and $U(p_3)$ and the smoothed points of global order $j+1$ which may be involved in a next diffeotopic perestroika is deduced

Besides, along the path S, as the curve $\mathcal{H}_m$ experiences various diffeotopic perestroikas $J(\alpha)$ and $J(\beta)$ increases or decreases. It follows from the Lemma 2.5 of Chapter 5 that imaginary points involved in a perestroika π belong to neighborhoods $U(p_i)$, $1 \le i \le 4$, in such a way that diffeotopic perestroikas in neighborhoods of global order j provide or smooth the imaginary points involved in diffeotopic perestroika in neighborhoods $\tilde{U}(p)$ of order $j+1$.

2)Consider now π a strong triple diffeotopic perestroika.

From the Lemma 2.6 of Chapter 5, strong triple point smoothed diffeotopic perestroika π requires a neighborhood $\tilde{U}(p)$ which contains a crossing a associated to a triple-point or a double-point of the curve $\mathcal{H}_m$. It is an easy consequence of the Lemma 2.6 of Chapter 5, that, according to our notations, the k-uple ($k \in \{2, 3\}$) is of the form (p_i, p_{i+1}, p_{i+2}), i even (resp, (p_i, p_{i+1}), i even) with $p_i \in U(p_1)$ belongs to the neighborhood $\tilde{U}(p_1)$ of global order i with $i+1 \le j \le m-1$. As already noticed in the proof of Lemma 2.6 of Chapter 5, the restriction of 3-uple

(p_i, p_{i+1}, p_{i+2}) to (p_i, p_{i+1}) is sufficient to define π; relative position of real branches involved in the topological meaning of p_{i+2} under π follows by induction.
Besides,the diffeotopic perestroika π is possible only if orientation and relative location of the real branches in the neighborhoods $U(p_1)$, $U(p_2)$, $U(p_3)$, $U(p_4)$ is one of the two required (see section 5 of Chapter 5). It follows from the Lemma 2.6 of Chapter 5 except around real branches involved in π, the situation remains the same inside $\cup_{1\leq i\leq 4}\tilde{U}(p)$ under diffeotopic perestroika.

In such a way, at the end of the step j of the lifting, we get an irreducible curve of degree j and type I inside $\rho^j(T_j \times U^2_{\mathbf{C}})$. This irreducible curve of degree j has no real double-point singularity but may have an even number of imaginary double-point singularity. The step $j+1$ of the lifting can be interpreted as the perturbation of the union of this curve of degree j with a line -namely, according to the Lemma 2.1 of Chapter 5 and its proof as the way to move the union of the g_j 1-handles of $\mathbf{C}\mathcal{H}_j \approx S^2_{g_j}$ with the $(j-1)$ more 1-handles of $\mathbf{C}\mathcal{H}_{j+1} \approx S^2_{g_j+(j-1)}$-.

We have considered only neighborhoods $\tilde{U}(p)$ of global order j, $2 \leq j \leq m-1$, and $\sharp\mathcal{P}_m - g_m = m-1$. Nonetheless, the lifting of $\tilde{h} : \mathbf{R}\tilde{\mathcal{H}}_m \to \mathbf{R}\tilde{\mathcal{A}}_m$ is entirely defined by induction on the global order, and we get the topological pair $(\mathbf{C}P^2, \mathbf{C}\mathcal{A}_m)$ from the method proposed. Indeed, consider the $m-2$ ovals $\mathcal{O}$ which intersect two neighborhoods $\tilde{U}(p)$ of global order m, and one neighborhood $\tilde{U}(p)$ of global order $m-1$. (The union of these $m-2$ ovals intersects any neighborhood $\tilde{U}(p)$, $p \in \mathcal{P}_m$ of global order m.) Any of these m-2 ovals $\mathcal{O}$ is involved in a diffeotopic perestroika implied (in the chain of diffeotopic perestroika) by a diffeotopic perestroika on neighborhood of order $m-1$. Hence, the lifting of $\tilde{h} : \mathbf{R}\tilde{\mathcal{H}}_m \to \mathbf{R}\tilde{\mathcal{A}}_m$ follows by induction.

Consequently, we get description of $\mathbf{C}\mathcal{A}_m$ in any neighborhood $\tilde{U}(p)$, $p \in \mathcal{P}_m$ and thus in $\cup_{p\in\mathcal{P}_m}\tilde{U}(p) = \mathbf{C}P^2$.

According to Lemma 2.1 of Chapter 5 and Lemma 2.3 of Chapter 5, we have defined a lifting of the path $\tilde{h} : \mathbf{R}\tilde{\mathcal{H}}_m \to \mathbf{R}\tilde{\mathcal{A}}_m$ in the space of generic immersion of the space of immersions of a circle into the plane to $S : \mathbf{R}\mathcal{H}_m \to \mathbf{R}\mathcal{A}_m$ which is the track on $\mathbf{R}P^2$ of a diffeotopy h_t of $\mathbf{C}P^2$, $t \in [0,1]$, $h(0) = \mathbf{C}\mathcal{H}_m$, $h(1) = \mathbf{C}\mathcal{A}_m$.
Q.E.D

Description of pairs $(\mathbf{C}P^2, \mathbf{C}\mathcal{A}_m)$ up to conj-equivariant isotopy. In this section, we shall first give a combinatorial method to describe pairs $(\mathbf{C}P^2, \mathbf{C}\mathcal{A}_m)$, up to conj-equivariant isotopy, where $\mathcal{A}_m$ is a smooth curve of degree m and type I. Then, we extend in Theorem 2.9 of Chapter 5 the description of the pair $(\mathbf{C}P^2, \mathbf{C}\mathcal{H}_m)$ stated in Theorem 3.9 of Chapter 2 for Harnack curve to any smooth curve of type I.

Combinatorial Description of pairs $(\mathbf{C}P^2, \mathbf{C}\mathcal{A}_m)$

In this part, we shall detail the method given in the third part of the proof of the Proposition 2.4 of Chapter 5 and describe the family of moves $\phi : \mathcal{H}_m \to \mathcal{S}_{I,m}$ where $\mathcal{S}_{I,m} = \mathcal{C}_{I,m}\backslash\mathcal{D}_m$.

For any curve $\mathcal{A}_m$ of degree m and type I, we have defined a lifting of the path $\tilde{h} : \mathbf{R}\tilde{\mathcal{H}}_m \to \mathbf{R}\tilde{\mathcal{A}}_m$ which is the track S on $\mathbf{R}P^2$ of a diffeotopy h_t of $\mathbf{C}P^2$, $t \in [0,1]$, $h(0) = \mathbf{C}\mathcal{H}_m$, $h(1) = \mathbf{C}\mathcal{A}_m$.

(Obviously, the path is not necessarily lifted continuously in the space of real algebraic curves in the sense that we can not affirm that any immersion of the circle into $\mathbf{R}^2$ with index g_m along an arbitrary path from $\mathbf{R}\tilde{\mathcal{H}}_m$ to $\mathbf{R}\tilde{\mathcal{A}}_m$ lift to a curve of degree m and type I.)

In this part, we shall describe the family of moves $\phi : \mathcal{H}_m \to \mathcal{S}_{I,m}$

We shall work with the terminology introduced in section 2 of Chapter 2, (see Proposition 3.7 of Chapter 2, and also Theorem 3.9 of Chapter 2 and consider the set of points $\cup_{i\in I} a_i = \cup_{n=1}^{m} A_n$ where A_n denotes the set of points perturbed in a maximal simple of deformation of $\mathcal{H}_n$. Recall that $A_{2k} = \{a_1, ..., a_{2k-1}, ..., a_{4k-2}\}$ where $a_1, ...a_{2k-1}$ are the crossings of $\mathcal{H}_{2k-1} \cup L$ and $a_{2k}, ..., a_{4k-2}$ are the crossings of $\mathcal{H}_{2k;1}$ where $\mathcal{H}_{2k;t}$, $t \in [0, 1]$ is a maximal simple deformation of $\mathcal{H}_{2k}$; $A_{2k+1} = \{a_1, ..., a_{2k}\}$ where $a_1, ...a_{2k}$ are the crossings of $\mathcal{H}_{2k} \cup L$. In such a way, we assign to the set $A2k = \{a_1, ..., a_{2k-1}, ..., a_{4k-2}\}$ the global order $2k$; and to the set $A_{2k+1} = \{a_1, ..., a_{2k}\}$ the global order $2k + 1$.

Fold the real point set of $\mathcal{H}_m$ in such a way for any $1 < j < m$, the two real points of $\mathcal{H}_{j+1}$ resulting from the perturbation of one crossing of $\mathcal{H}_j \cup L$ are glued each other in the initial crossing. Then, fold the curve $\mathcal{H}_m$ in any other points perturbed in its construction; namely, glue the two branches involved in the local topological meaning of any critical point a_i, $2k \le i \le 4k-2$ of Harnack polynomials B_{2k}, $2k \le m$.

The curve $\mathcal{A}_m$ results from desingularization of all these points with multiplicity 2 compatible with the way S goes through the discriminant hypersurface. It is equivalent to unfold in a way different from the one given $\mathcal{H}_m$. Restriction on the unfolding is given by the set of smoothed diffeotopic perestroika experienced along the path S from $\mathbf{R}\mathcal{H}_m$ to $\mathbf{R}\mathcal{A}_m$. Along the path only triple-point with imaginary branches, strong triple point, inverse and direct self-tangency diffeotopic perestroika are encountered.

Moreover, according to the third part of the proof of the Proposition 2.4 of Chapter 5,

(1) desingularization (unfolding) are done in increasing global order on the set of points perturbed in the construction of $\mathcal{H}_m$. Locally, any choice of a desingularization is compatible with one diffeotopic perestroika or lets fix the real components of $\mathcal{H}_m$.
(2) Imaginary points are involved in one diffeotopic perestroika π along the path S in such a way that if $J_\pi(\gamma) = 2$, $\gamma \in \{\alpha, \beta\}$, then it provides imaginary points involved in -at least one and at most two- next diffeotopic perestroikas π' along S such that $J_{\pi'}(\gamma) = -2$.
From the Lemma 2.5 of Chapter 5, any real or imaginary point involved in a diffeotopic perestroika belongs to a neighborhood $\tilde{U}(p) \supset U(p) = \rho^m(D(p) \times U^2_{\mathbf{C}})$, $p \in \mathcal{P}_m$, and the next perestroikas are provided by move of real branches in $U(p)$. Consequently, in the chain of moves, imaginary singular points are smoothed in such a way a the end of the procedure one gets a curve without singular points.

Let us summarize properties of the path $S : [0, 1] \to \mathbf{R}\mathcal{C}_m$ in the following Theorem 2.7 of Chapter 5.

Recall that given the triangle T_m, the real projective space $\mathbf{R}P^2$ may be deduced from the square T_m^* made of T_m and it symmetric copies $T_{m,x} = s_x(T_m)$, $T_{m,y} = s_y(T_m)$ $T_{m,xy} = s(T_m)$ where $s_x, s_y, s = s_x \circ s_y$ are reflections with respect to the coordinates axes. Given a triangle T of the set T_m,$T_{m,x}$, $T_{m,y}$, $T_{m,xy}$ $s(T)$ denotes the symmetric copy of T.

THEOREM 2.7. *Let $\mathcal{A}_m$ be a curve of degree m and type I. Let $\mathcal{H}_m$ be the Harnack curve of degree m. Then, up to conj-equivariant isotopy of $\mathbf{C}P^2$, there exists a path $S : [0,1] \to \mathbf{R}\mathcal{C}_m$ with $S(0) = \mathbf{R}\mathcal{H}_m$, $S(1) = \mathbf{R}\mathcal{A}_m$ which crosses the discriminant hypersurface $\mathbf{R}\mathcal{D}_m$ in curves with crossings, real points of intersection of a real branch and two conjugate imaginary branches, strong triple-points, direct and inverse self-tangency points.*

Curves along S may be deduced one from another from smoothed diffeoopic perestroikas with the property that any smoothed diffeotopic perestroika along S defines double-points involved in next smoothed diffeotopic perestroikas along S.

Furthermore,

(1) *any real ordinary double point of curves along S is a crossing which belongs to the set of points perturbed in the construction of $\mathcal{H}_m$*
(2) *any double-point involved in a diffeotopic perestroika is a crossing which belongs to the set of points perturbed in the whole construction of $\mathcal{H}_m$.*

Besides, smoothed diffeotopic perestroikas may be described in the patchworking scheme as follows: Let $T \in \{T_m, T_{m,x}, T_{m,y}, T_{m,xy}\}$;

(1) *inverse self tangency diffeotopic perestroika in the positive direction acts as follows: an oval $\mathcal{O}$ of $\mathcal{H}_m$ contained in a triangle T becomes an oval $s(\mathcal{O})$ of the symmetric triangle $s(T)$,*

    in the negative direction, according to the definition of the self tangency diffeotopic perestroika $s(\mathcal{O})$ disappears in two real branches with opposite orientation.

    Moreover, in the positive (resp negative) direction, it requires (resp, smooths) imaginary points provided by a triple point with imaginary diffeotopic perestroika in the negative direction.
(2) *direct self tangency diffeotopic perestroika acts as follows: it does not change the real part but in the negative direction it provides the imaginary points of a triple-point diffeotopic perestroika with imaginary conjugate branches in the positive direction;*

    in the positive direction it smooths the imaginary point of a triple-point diffeotopic perestroika with imaginary conjugate branches in the negative direction.
(3) *triple-point with imaginary conjugate branches diffeotopic perestroika is as follows:*

    an oval $\mathcal{O}$ of $\mathcal{H}_m$ contained in a triangle T becomes an oval $s(\mathcal{O})$ of the symmetric triangle $s(T)$.
(4) *strong triple point perestroika may be experienced only in an open which contains a k-point (in case m even : $k = 3$; in case m odd: $k = 3$ or $k = 2$) of $\mathcal{H}_m$.*

REMARK 2.8. (1) One can easily deduce the maximal nest curve of degree $m = 2k$ (which are L-curves and thus have standard Arnold surfaces) from the Harnack curve $\mathcal{H}_{2k}$.
Indeed, choose an orientation on the recursive line $\mathbf{R}l_{2k-j}$ opposite to the one chosen in the construction of $\mathcal{H}_m$. A slight perturbation of the resulting curves given by standard modification on real double point singularity compatible with the orientation of the resulting curve leads to the maximal nest curve.
Besides, according to the Theorem 2.7 of Chapter 5, there exists a path from $\mathcal{H}_{2k}$ to the maximal nest curves $< 1 >^k$. Such path may be described by smoothed diffeotopic perestroika, some of which (for $k > 2$) are smoothed strong triple point diffeotopic perestroikas.
(2) Obviously, from the method summarized in theorem 2.7 of Chapter 5, one can deduce whether there exist curves of a degree m and type I with a given real scheme.
For example, the non-existence of the curve of degree 7 and real scheme $J \cup < 1 < 14 >>$ is easy to deduce. Indeed, it can be easily proved that the existence of such curve is in contradiction with the possible situations inside the three neighborhoods $U(p)$, $p \in \mathcal{P}_7$ of global order 6 which contain crossings associated to 2-uple. (More generally, one can notice that the $2k-3$ 2-uples of $\mathcal{H}_{2k+1}$ define possible intersection of the one-side component of $\mathcal{H}_{2k+1}$ with the boundary of the Möbius band of $\mathbf{R}P^2$ embedded in $\mathbf{C}P^2$.)

Description of pairs $(\mathbf{C}P^2, \mathbf{C}\mathcal{A}_m)$ up to conj-equivariant isotopy

We shall now in Theorem 2.9 of Chapter 5 extend the results of Theorem 3.9 of Chapter 2 to any smooth curve of type I.
According to Theorem 3.9 of Chapter 2 there exists a finite number I of 4-balls $B(a_i)$ globally invariant by complex conjugation; such that, up to conj-equivariant isotopy of $\mathbf{C}P^2$ $\mathcal{H}_m \setminus \cup_{i\in I} B(a_i) = \cup_{i=1}^m L_i \setminus \cup_{i=1}^I B(a_i)$; inside any $B(a_i)$ $\mathcal{H}_m$ is the perturbation of crossing centered the in point a_i. The following statement is a consequence of the definition of $S : [0,1] \to \mathbf{R}\mathcal{C}_m$ given in Proposition 2.4 of Chapter 5 and the fact that any real singular point which participated in diffeotopic perestroika along S belongs to $\cup_{i=1}^I a_i$.

THEOREM 2.9. *Let $\mathcal{A}_m$ be a curve of degree m and type I. There exists a finite number I ($I = 1 + 2...m + \Sigma_{k=2}^{k=[m/2]} 2k-3$) of disjoints 4-balls $B(a_i)$ invariant by complex conjugation and centered in points a_i of $\mathbf{R}P^2$ such that, up to conj-equivariant isotopy of $\mathbf{C}P^2$:*

(1) *$\mathcal{A}_m \setminus \cup_{i\in I} B(a_i) = \cup_{i=1}^m L_i \setminus \cup_{i\in I} B(a_i)$ where L_1,...,L_m are m projective lines with $L_i \setminus \cup_{i\in I} B(a_i) \cap L_j \setminus \cup_{i\in I} B(a_i) = \emptyset$ for any $i \neq j$,$1 \leq i,j \leq m$.*
(2) *situations inside 4-balls $B(a_i)$ are covered by perturbation of type 1 or type 2 of the crossing a_i.*

proof:

Our proof is based on the proof of Proposition 2.4 of Chapter 5, Theorem 2.7 of Chapter 5 and Theorem 3.9 of Chapter 2.

According to Theorem 3.9 of Chapter 2, there exists a finite number I of disjoints 4-balls $B(a_i)$ invariant by complex conjugation and centered in points a_i of $\mathbf{R}P^2$ such that , up to conj-equivariant isotopy of $\mathbf{C}P^2$:

(1) $\mathcal{H}_m \setminus \cup_{i \in I} B(a_i) = \cup_{i=1}^m L_i \setminus \cup_{i \in I} B(a_i)$ where L_1,...,L_m are m projective lines with $L_i \setminus \cup_{i \in I} B(a_i) \cap L_j \setminus \cup_{i \in I} B(a_i) = \emptyset$ for any $i \neq j$,$1 \leq i, j \leq m$.
(2) situations inside 4-balls $B(a_i)$ are perturbations of type 1 of crossing.

Let $\mathbf{C}\mathcal{H}_m^+$ be the half of $\mathbf{R}\mathcal{H}_m$ which induces orientation on the real part of $\mathbf{R}\mathcal{H}_m$. The conj-equivariant isotopy brings $\mathbf{C}\mathcal{H}_m^+ \setminus \cup_{i=1}^I B(a_i)$ to halves $\mathbf{C}L_i^+ \setminus \cup_{i=1}^I B(a_i)$ of lines $L_i \setminus \cup_{i=1}^I B(a_i)$, $1 \leq i \leq m$, which induce an orientation on the real part $\mathbf{R}L_i$.

The path $S : [0,1] \to \mathbf{R}\mathcal{C}_{m,I}$ $S(0) = \mathbf{C}\mathcal{H}_m$, $S(1) = \mathbf{C}\mathcal{A}_m$ can be seen as the track on $\mathbf{R}P^2$ of a diffeotopy h_t of $\mathbf{C}P^2$, $t \in [0,1]$, $h(0) = \mathbf{C}\mathcal{H}_m$, $h(1) = \mathbf{C}\mathcal{A}_m$ with $h_t(\mathbf{C}\mathcal{H}_m) \cap U(p) \subset U(p)$ and may be described as a family moves defined by smoothed diffeotopic perestroikas.

According to the proof of proposition 2.4 of Chapter 5, any real double point involved in a diffeotopic perestroika belongs to $\cup_{i \in I} B(a_i)$.

Besides, nonetheless relative location of any part of $\mathcal{H}_m \setminus \cup_{i \in I} B(a_i)$ conj-equivariant isotopic to a part $L_i \setminus \cup_{i \in I} B(a_i)$ is changed under a diffeotopic perestroika, its orientation remains the same before and after diffeotopic perestroika.

Therefore, for any line L_i, the half $\mathbf{C}L_i^+ \setminus \cup_{i \in I} B(a_i)$ of $L_i \setminus \cup_{i \in I} B(a_i)$ which induces orientation on $\mathbf{R}L_i \setminus \cup_{i \in I} B(a_i)$ is up to conj-equivariant isotopy a part of the half $\mathbf{C}\mathcal{A}_m^+$ of $\mathbf{C}\mathcal{A}_m$ which induces orientation on $\mathbf{R}\mathcal{A}_m$.

Hence, outside $\cup_{i=1}^I B(a_i)$, the curve $\mathcal{A}_m$ is union of m lines minus their intersection with $\cup_{i=1}^I B(a_i)$.

Moreover, situations inside 4-balls $B(a_i)$ are covered by two cases which are locally perturbations of type 1 or type 2 of a crossing.

Q.E.D

3. Construction of Curves of type II

As already known, if an M-curve is constructed curves with fewer components are easily constructed. Besides, M-curves, as curves of type I, have been studied in the section 2 of Chapter 5. We shall now introduce a method inspired from the preceding one, which will give all $M-i$-curves of degree m and therefore any curves of type II from the Harnack curve $\mathcal{H}_m$.

In the previous section, it was essential that all curves have orientable real point set. In order to consider curves with non-orientable real set of points we shall modify the argument introduced in the preceding part. We shall call *perturbation in the non-coherent direction of a crossing* of a curve the desingularization which induces non-orientability of the real part of the resulting curve.

For a generic curve $\mathcal{A}_m$ of degree m and genre g, $0 < g < \frac{(m-1)(m-2)}{2}$, by *smoothing in the coherent direction* of one double point of its real point set $\mathbf{R}\mathcal{A}_m$, we shall understand the Morse modification in $\mathbf{R}^2$ in the direction coherent to a complex orientation of $\mathcal{A}_m$, by *smoothing in the non-coherent direction* of one double point of its real point set $\mathbf{R}\mathcal{A}_m$, we shall understand the Morse modification in $\mathbf{R}^2$ at the double point in the direction non-coherent to a complex orientation

of $\mathcal{A}_m$, i.e not compatible with an orientation of the real point set of the resulting curve.

Let us start our study of curves of type II by a Lemma analogous to the Lemma 2.1 of Chapter 5 stated for curves of type I.

Call *smoothed generic immersion of the circle* S^1 *into the plane* $\mathbf{R}^2$, the smooth submanifold of $\mathbf{R}P^2$ deduced from the generic immersion of the circle by modification at each real double point which is either a Morse modification in $\mathbf{R}^2$ coherent or non-coherent to the complex orientation, or the Morse modification in $\mathbf{R}P^2$ which associates to the double point of $\mathbf{R}^2$ two points of the line at infinity of $\mathbf{R}P^2$.

A generalization of Whitney's theorem to the case of real algebraic curves will be provided by the following Lemma:

LEMMA 3.1. *Let* $\mathcal{A}_m$ *be a smooth curve of degree* m *and type* II *with non-empty real points set* $\mathbf{R}\mathcal{A}_m$ *then* $\mathbf{R}\mathcal{A}_m \subset \mathbf{R}P^2$ *is a smoothed generic immersion of the circle* S^1 *into the plane* $\mathbf{R}^2$ *with* $\frac{(m-1)(m-2)}{2} \leq n \leq \frac{(m-1)(m-2)}{2} + [\frac{m}{2}]$, *double points and Whitney index* $\frac{(m-1)(m-2)}{2} + 1$. *The smoothing is such that* $\frac{(m-1)(m-2)}{2}$ *double points are smoothed in* $\mathbf{R}^2$ *by Morse modification and at least one Morse modification is in the non coherent direction.*

REMARK 3.2. One can associate to any smooth curve of type II a three-dimensional rooted tree.

proof:

Our proof makes use of properties of the complex point set $\mathbf{C}\mathcal{A}_m$ embedded in $\mathbf{C}P^2$. It is deduced from an argument similar to the one given for curves of type I.

Consider the usual handlebody decomposition of $\mathbf{C}P^2 = B_0 \cup B_1 \cup B$ where B_0,B_1,B are respectively 0,2 and 4 handles.

The ball B_0 and B_1 meet along an unknotted solid torus $S^1 \times B^2$. The gluing diffeomorphism $S^1 \times B^2 \to S^1 \times B^2$ is given by the $+1$ framing map. In such a way, the canonical $\mathbf{R}P^2$ can be seen as the union of a Möbius band $\mathcal{M}$ and the disc $D^2 \subset B$ glued along their boundary. The Möbius band $\mathcal{M}$ lies in $B_0 \cap B_1 \approx S^1 \times D^2$ with $\partial\mathcal{M}$ as the $(2,1)$ torus knot, and $D^2 \subset B$ as the properly imbedded unknotted disc. The complex conjugation switches B_0 and B_1 and lets fix $\mathcal{M}$, it rotates B around D^2.

The set of complex points of $\mathcal{A}_m$ is an orientable surface of genus $g_m = \frac{(m-1)(m-2)}{2}$, i.e it is diffeomorphic to a sphere S^2 with g_m 1-handles $S^2_{g_m}$. We shall denote h the diffeomorphism $h : \mathbf{C}\mathcal{A}_m \to S^2_{g_m} \subset \mathbf{R}^3$.

Assume S^2 provided with a complex conjugation with fixed point set a circle S^1 which divides the sphere S^2 into two halves.

Without loss of generality, one can assume that any disc removed from S^2 and then closed up to give $S^2_{g_m}$ intersects S^1 and the two halves of S^2. In such a way, $\mathbf{R}\mathcal{A}_m \subset \mathbf{R}P^2$ intersects each 1-handle.

Fix D^2 the two disc of $\mathbf{R}P^2 = \mathcal{M} \cup D^2$ in such a way that the boundary circle of D^2 is S^1 and therefore each one handle belongs to the solid torus $S^1 \times D^2 \supset \mathcal{M}$.

Since the interior $(D^2)^0$ of D^2 is homeomorphic to $\mathbf{R}^2$, one can project $\mathbf{R}\mathcal{A}_m$ (up to homeomorphism $\mathbf{R}^2 \approx (D^2)^0$) in a direction perpendicular to $\mathbf{R}^2$ onto $\mathbf{R}^2$. We may suppose that the direction of the projection is generic i.e all points of self-intersection of the image on $\mathbf{R}^2$ are double and the angles of intersection are non-zero.

Let $r : \mathbf{R}^3 \to \mathbf{R}^2$ be the projection which maps $h(\mathbf{R}\mathcal{A}_m) \subset \mathbf{R}^3$ to $\mathbf{R}^2$. Consider an oriented tubular fibration $N \to \mathbf{C}\mathcal{A}_m$. Since $\mathbf{C}\mathcal{A}_m$ is diffeomorphic to a sphere S^2 with g_m 1-handles, one can consider the restriction of $\mathbf{C}\mathcal{A}_m$ diffeomorphic to the torus T^2 given by the sphere S^2 with one of g_m 1-handle.

The oriented tubular neighborhood of $h(\mathbf{C}\mathcal{A}_m) \cap T^2$, as oriented tubular neighborhood of the torus T^2, intersects the solid torus
$S^1 \times D^2 \supset \mathcal{M}$ with $\partial M \subset S^1 \times S^1$ as the $(2, 1)$ torus knot.

Hence, since in $\mathbf{C}P^2$, each real line is split by its real part into two halves lines conjugate to each others, and 2 disjoint circles always divide the torus, one can assume, without loss of generality, that the real part of the restriction of $h(\mathbf{C}\mathcal{A}_m) \cap T^2$ belongs to the boundary of the Möbius band in such a way that its projection to $\mathbf{R}^2$ gives one crossing.

Besides, some double-points of $r(h(\mathbf{R}\mathcal{H}_m))$ may result either from two points of $\mathbf{R}\mathcal{A}_m$ which belong to two different handles or one point which belongs to a 1-handle and the other point belongs to S^2. It easy to see that, in both cases, the projection leads to an even number of such double-points.

Hence, from an argumentation similar to the one given in case of curves of type I, it follows that $\mathbf{R}\mathcal{A}_m$ is the smoothed immersion of a generic immersion of the circle S^1 into the plane $\mathbf{R}^2$ with n, $\frac{(m-1)(m-2)}{2} \leq n \leq \frac{(m-1)(m-2)}{2} + [\frac{m}{2}]$, double points and Whitney index $\frac{(m-1)(m-2)}{2} + 1$.

Besides, it is obvious, since $\tilde{\mathcal{A}}_m$ is of type II, that at least one-double point of $r(h(\mathbf{R}\tilde{\mathcal{A}}_m))$ is smoothed in the non-coherent direction (in other words, at least two circles of the $g_m + 1$ which divide the surface $S^2_{g_m}$ glue and disappear in one circle.) Q.E.D.

Given a generic immersion of the circle S^1 into the plane $\mathbf{R}^2$ and $\mathcal{S}$ the set of its singular points. We shall call *partially smoothed* generic immersion of the circle S^1 into the plane $\mathbf{R}^2$ the singular real part deduced from the generic immersion of the circle by modification of a set $\mathcal{K} \subset \mathcal{S}$, $\mathcal{K} \neq \mathcal{S}$ of real double points.

The following Lemma, analogue to the Lemma 2.3 of Chapter 5 of the previous part, enlarges the preceding statement to singular curves.

LEMMA 3.3. *Let $\mathcal{A}_m$ be a singular curve of degree m and type II with non-empty real points set $\mathbf{R}\mathcal{A}_m$ and non-degenerate singular points, then its real point set $\mathbf{R}\mathcal{A}_m \subset \mathbf{R}P^2$ is a partially smoothed generic immersion of the circle S^1 into the plane $\mathbf{R}^2$ with $n \geq \frac{(m-1)(m-2)}{2}$ double points and Whitney index $\frac{(m-1)(m-2)}{2} + 1$ if and only if its set of singular points consists of at most $\frac{(m-1)(m-2)}{2}$ crossings and at least one crossing is smoothed in $\mathbf{R}^2$ in the non-coherent direction.*

proof:

It follows from an argument similar to the one used in the proof of Lemma 3.1 of Chapter 5. Q.E.D

Although the stratification of the set singular curves in the variety $\mathcal{C}_{I;g}$ may not be extended to curves of type II, we shall lift the path provided by Whitney's theorem and Lemma 3.1 of Chapter 5 to algebraic curves of type II.

As previously, we shall consider the space $\mathbf{R}\mathcal{C}_m$ of real algebraic curves of degree m and its subset $\mathbf{R}\mathcal{D}_m$ constituted by real singular algebraic curves of degree m. The set $\mathbf{R}\mathcal{D}_m$ has an open every dense subset which consist of curves with only one singular point which is a non-degenerate double point (i.e a solitary real double-point, or a crossing). This subset is called *principal part* of the set $\mathbf{R}\mathcal{D}_m$. A generic path in $\mathbf{R}\mathcal{C}_m$ intersects $\mathbf{R}\mathcal{D}_m$ only in its principal part and only transversally. When a singular curve occurs in a generic one-parameter family curves, the moving curve is passing through a Morse modification.

Here are simple properties of Morse modifications which motivate our study:

(1) Under a Morse modification a curve of type I can turn only to a curve of type II and the number of its real components decreases.
(2) A complex orientation of a non-singular curve turns into an orientation of the singular curve which appears at the moment of the modification: the orientations defined by complex orientation of the curve on the two arcs which approach each other and merge should be coherent with an orientation of the singular curve.

We shall generalize the previous method giving any smooth curve $\mathcal{A}_m$ of type I to smooth curves of type II and deduce curves of type II from smoothing (including smoothing in a direction non-coherent to a complex orientation) generic curves of type I, genus g and type I.

Assume $\mathcal{H}_m$ the Harnack of degree m obtained via the patchworking method. For any point $p \in \mathcal{P}_m$, the subset $U(p) = \rho^m(D(p, \epsilon) \times U^2_{\mathbf{C}}) \subset (\mathbf{C}^*)^2$ intersects $(\mathbf{R}^*)^2$ in four discs. We shall denote by $\mathcal{B}_m$ the set constituted by the union of the four 2-discs of $U(p) \cap (\mathbf{R}^*)^2$ taken over the points $p \in \mathcal{P}_m$.

PROPOSITION 3.4. *Let $\mathcal{H}_m$ be the Harnack curve of degree m and $\mathcal{A}_m$ be a smooth curve of type II. Then, up to conj-equivariant isotopy of $\mathbf{C}P^2$, there exists a path*

$$S : [0, 1] \to \mathbf{R}\mathcal{C}_m$$

$S(0) = \mathbf{R}\mathcal{H}_m$, $S(1) = \mathbf{R}\mathcal{A}_m$ which intersects the discriminant hypersurface $\mathbf{R}\mathcal{D}_m$ in such a way that:

(1) *any real ordinary double point of curves along S is a crossing which belongs to a real 2-disc of $\mathcal{B}_m$.*
(2) *the curve $\mathcal{A}_m$ is deduced from smoothing the real part of a generic curve of degree m of which double points are crossings. At least one point is smoothed in a direction non-coherent with a complex orientation of the real part of the curve.*

proof:

As already done in case of curves of type I, using the path provided by Lemma 3.1 of Chapter 5 and Whitney's theorem, according to the Lemma 3.3 of Chapter 5, and the fact that $\mathbf{C}\mathcal{H}_m$ and $\mathbf{C}\mathcal{A}_m$ are diffeotopic, we shall characterize the track S on $\mathbf{R}P^2$ of a diffeotopy h_t of $\mathbf{C}P^2$, $t \in [0, 1]$, $h(0) = \mathbf{C}\mathcal{H}_m$, $h(1) = \mathbf{C}\mathcal{A}_m$.

Method which provides $(\mathbf{C}P^2, \mathbf{C}\mathcal{A}_m)$ *from* $(\mathbf{C}P^2, \mathbf{C}\mathcal{H}_m)$.

Our method is a slightly modified version of the preceding one where any curve $\mathcal{A}_m$ of type I is obtained from $\mathcal{H}_m$.

We shall prove that up to modify the coefficients of the polynomial giving the curve $\mathcal{A}_m$, one can always assume that there exists a diffeotopy h_t of $\mathbf{C}P^2$ $h(0) = \mathbf{C}\mathcal{H}_m$, $h(1) = \mathbf{C}\mathcal{A}_m$ with the property $h_t(\mathbf{C}\mathcal{A}_m \cap U(p)) \subset U(p)$ for any $p \in \mathcal{P}_m$.

We shall refer to the method of section 2 of Chapter 5 and stress only the modification.

According to the Lemma 3.1 of Chapter 5, consider $\mathbf{R}\mathcal{H}_m$ and $\mathbf{R}\mathcal{A}_m$ as smoothed immersions of the circle into the plane with the same Whitney index. Denote $\mathbf{R}\tilde{\mathcal{H}}_m$, $\mathbf{R}\tilde{\mathcal{A}}_m$ the corresponding immersions. From the Whitney's theorem ([**20**]), there exists a path $\tilde{h}$ which connects $\mathbf{R}\tilde{\mathcal{H}}_m$ and $\mathbf{R}\tilde{\mathcal{A}}_m$. Thus, the definition of the path S is reduced to the definition of a lifting of the path which connects $\mathbf{R}\tilde{\mathcal{H}}_m$ and $\mathbf{R}\tilde{\mathcal{A}}_m$ in the space of immersions of the circle into $\mathbf{R}^2$ to a path in the space of curve of degree m and type II.

Define now the *smoothed perestroika* of a perestroika, the change obtained by smoothing the fragments involved in the perestroika where smoothing before perestroika are taken only in a direction coherent and smoothing after perestroika may be taken in a direction non-coherent.

Then, as in the proof of proposition 2.4 of Chapter 5 of section 2 of Chapter 5 where curves of type I were under consideration, since any smooth curve is irreducible, and any reducible polynomial is the product of a finite reducible one, according to Lemma 3.1 of Chapter 5 and Lemma 3.3 of Chapter 5, we may lift the path $\tilde{h}$ in the space of immersion of the circle into the plane from smoothed perestroikas in the space $\mathcal{C}_{I;g}$ of curve of degree $n \leq m$, type I and genus $0 \leq g \leq \frac{(n-1).(n-2)}{2}$.

In such a way, using the path provided by Lemma 3.1 of Chapter 5 and Whitney's theorem, according to the Lemma 3.3 of Chapter 5 and the fact that $\mathbf{C}\mathcal{H}_m$ and $\mathbf{C}\mathcal{A}_m$ are diffeotopic, we characterize the track S on $\mathbf{R}P^2$ of a diffeotopy h_t of $\mathbf{C}P^2$, $t \in [0,1]$, $h(0) = \mathbf{C}\mathcal{H}_m$, $h(1) = \mathbf{C}\mathcal{A}_m$.

Q.E.D

Remark 3.5. As already known, if two $M-2$-curves are obtained one from the other by a deformation through a double nondegenerate point, one of them is of type I and the other is of type II. Since, any $M-1$-curve is of type II, according to our method it remains to say that two deformations in the non-coherent direction may turn a curve of type I to a curve of type I.

As a curve of degree m moves as a point of $\mathbf{R}\mathcal{C}_m$ along an arc which has a transversal intersection with the principal part, then the set of real points of this curve undergoes either a Morse modification of index 0 or 2 (the curves acquires a solitary singular point, which then becomes a new oval, or else one of the ovals contracts to point) or Morse modification of index 0, an oval contracts to a point or a Morse modification of index 1 (two real arcs of the curve approach one other and merge, then diverge in their modified form.)

As already introduced, we denote A_{m+1} the set of crossings perturbed in the construction of $\mathcal{H}_{m+1}$ from $\mathcal{H}_m$ introduced in proposition 3.1 of Chapter 2.

In case $m+1=2k$, $A_{2k}=\{a_1,...,a_{2k-1},...,a_{4k-2}\}$ where $a_1,...a_{2k-1}$ are crossings of $\mathcal{H}_{2k-1}\cup L$ and $a_{2k},...,a_{4k-2}$ are crossings of $\tilde{X}_{2k;\tau}$.

In case $m+1=2k+1$, $A_{2k+1}=\{a_1,...,a_{2k}\}$ where $a_1,...a_{2k}$ are crossings of $\mathcal{H}_{2k}\cup L$.

We may characterize Morse modifications on a curves of degree m as follows:

COROLLARY 3.6. *Any Morse modification on a curve of degree m may be described, up to conj-equivariant isotopy, by perturbations of crossings of a subset of* $\cup_{n=1}^{m}A_m$

proof: It is a straightforward consequence of proposition 3.4 of Chapter 5. Q.E.D.

Description of pairs $(\mathbf{C}P^2, \mathbf{C}\mathcal{A}_m)$ up to conj-equivariant isotopy

THEOREM 3.7. *Let $\mathcal{A}_m$ be a curve of degree m and type II with non-empty real part. There exists a finite number I of disjoints 4-balls $B(a_i)$ invariant by complex conjugation and centered in points a_i of $\mathbf{R}P^2$ such that, up to conj-equivariant isotopy of $\mathbf{C}P^2$:*

(1) $\mathcal{A}_m \setminus \cup_{i\in I} B(a_i) = \cup_{i=1}^{m} L_i \setminus \cup_{i\in I} B(a_i)$ *where* $L_1,\ldots,L_m$ *are* m *projective lines with* $L_i \setminus \cup_{i\in I} B(a_i) \cap L_j \setminus \cup_{i\in I} B(a_i) = \emptyset$ *for any* $i \neq j, 1 \leq i, j \leq m$.
(2) *situations inside* 4*-balls* $B(a_i)$ *are covered by perturbation of type* 1 *or type* 2 *of the crossing* a_i.

proof:

Our proof is based on the previous method and the theorem 3.9 of Chapter 2. The argument is analogous to the one of the proof of theorem 2.9 of Chapter 5 where curves of type I were under consideration. Q.E.D

CHAPTER 6

Arnold surfaces of curves of even degree -Proof of the Rokhlin's conjecture-

In this section, we shall prove the Rokhlin's conjecture:

Theorem (Rokhlin's conjecture)

Arnold surfaces $\mathfrak{A}$ are standard for all curves of even degree with non-empty real part.

proof:

The proof is based on the Livingston's theorem and the following statement (deduced from theorem 2.9 of Chapter 5 and theorem 3.7 of Chapter 5)

Given $\mathcal{A}_m$ a curve of degree m and type I or type II with non-empty real part. There exists a finite number I of disjoints 4-balls $B(a_i)$ invariant by complex conjugation and centered in points a_i of $\mathbf{R}P^2$ such that up to conj-equivariant isotopy of $\mathbf{C}P^2$:

(1) $\mathcal{A}_m \setminus \cup_{i \in I} B(a_i) = \cup_{i=1}^{m} L_i \setminus \cup_{i \in I} B(a_i)$ where L_1,...,L_m are m projective lines with $L_i \setminus \cup_{i \in I} B(a_i) \cap L_j \setminus \cup_{i \in I} B(a_i) = \emptyset$ for any $i \neq j$,$1 \leq i, j \leq m$.
(2) situations inside any 4-balls $B(a_i)$ are covered by perturbation of type 1 or type 2 of the crossing a_i.

It follows from an argumentation similar to the one given in Chapter 3 (see theorem 0.10 of Chapter 3 where it is stated that Arnold surfaces of Harnack curves $\mathcal{H}_{2k}$ are standard surfaces in S^4) that any Arnold surface of a curve of even degree $2k$ with non-empty real part is standard.

Q.E.D

figure 1.1:perturbation of type 1

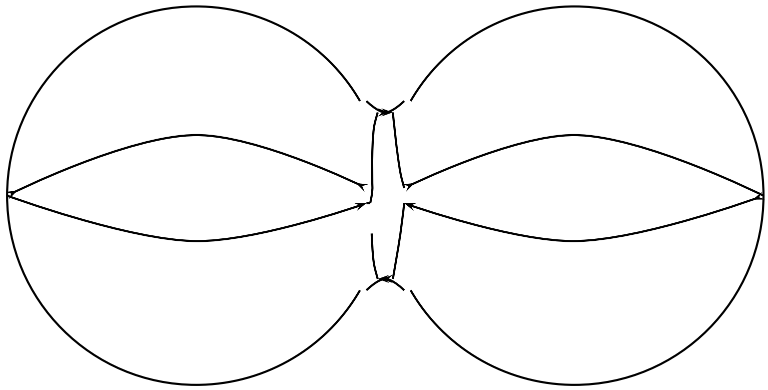


figure 1.2: perturbation of type 2

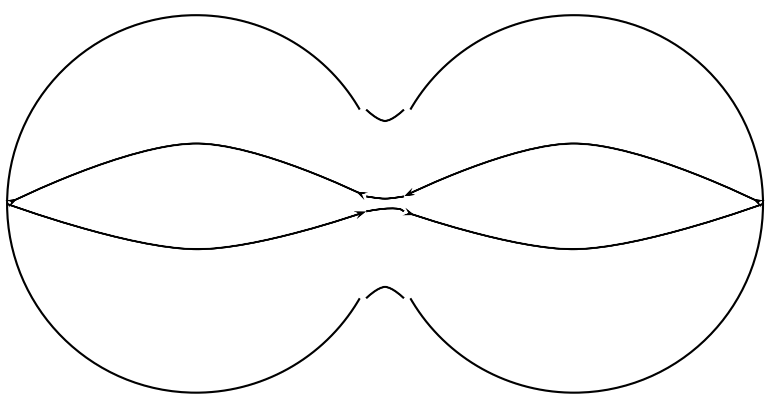


figure 3.1

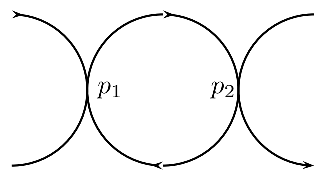
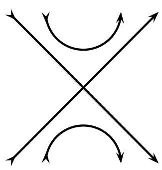


Figure 5.1-Smoothings



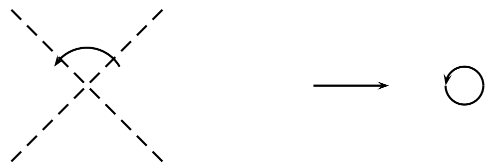


Figure 5.2-Weak triple point Perestroika

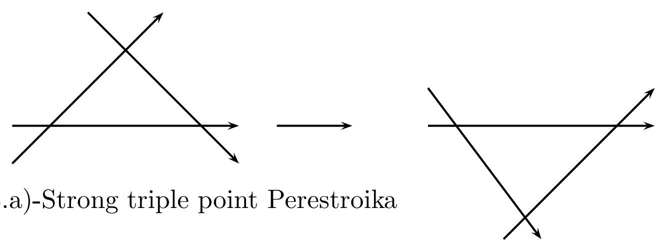


Figure 5.(3.a)-Strong triple point Perestroika

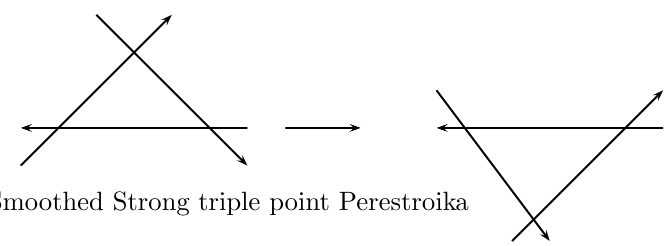


Figure 5.(3.b)-Smoothed Strong triple point Perestroika

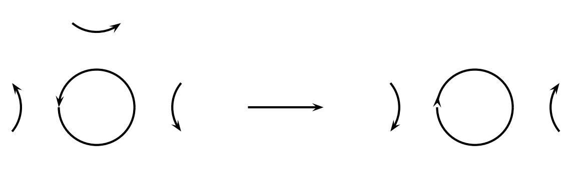


Figure 5.4-Smoothed real inverse tangency Perestroika

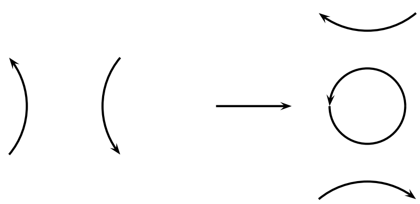
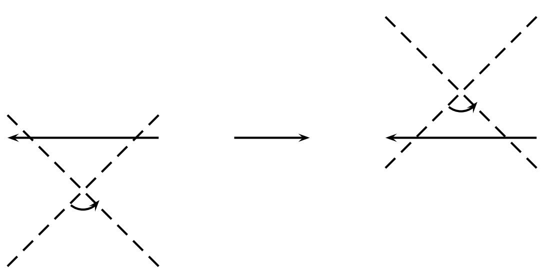
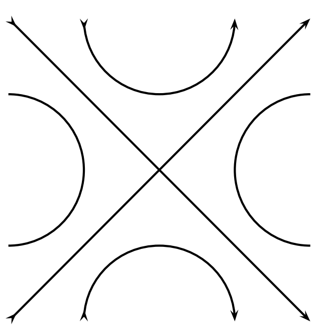


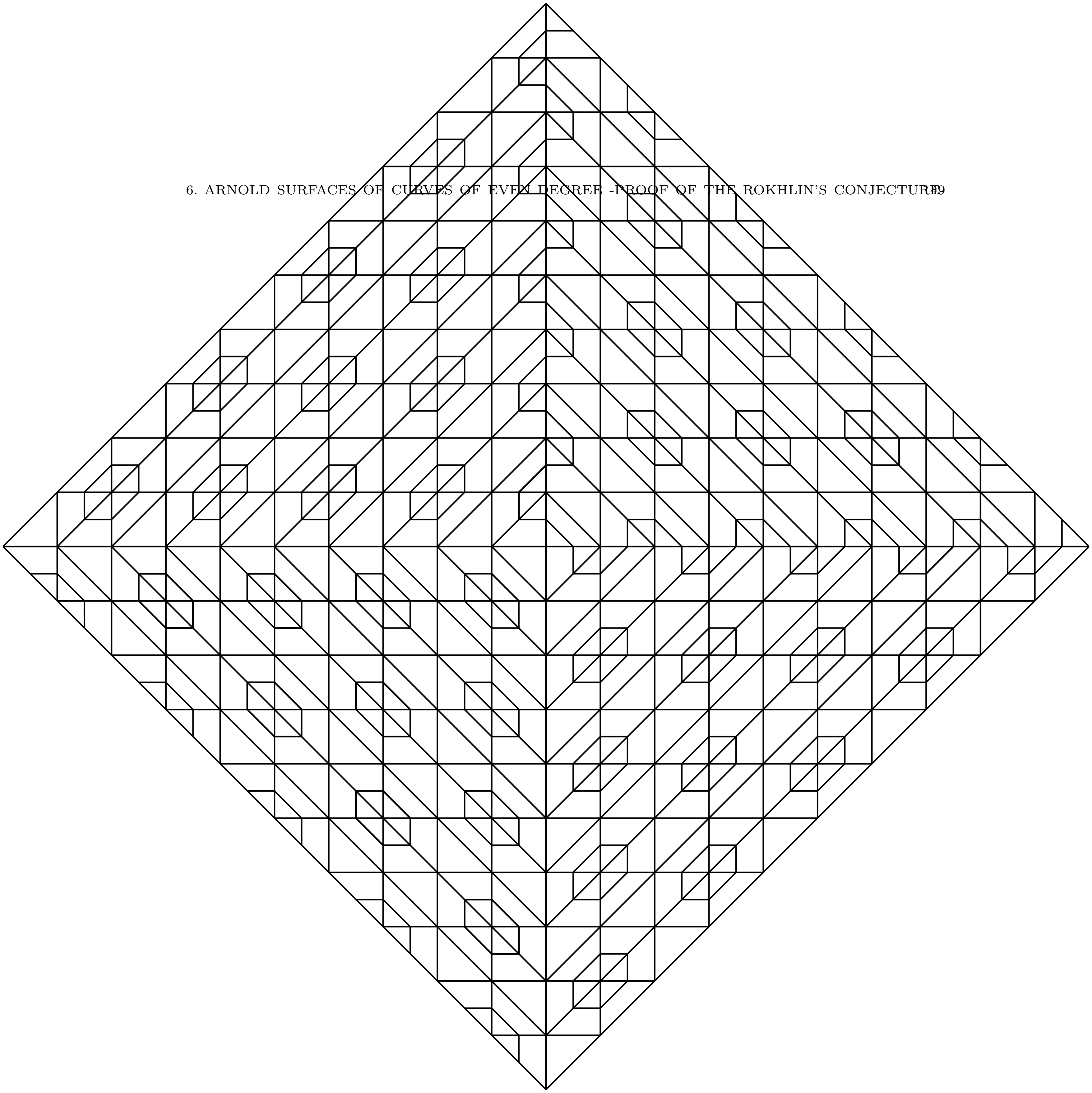
Figure 5.5-Triple-point with imaginary branches Perestroika



Morse modification in $\mathbf{R}^2$

in the direction coherent and non-coherent to a complex orientation





Harnack T-curve of degree 10

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